Mr. G's little booklet on

Trigonometrical (Circular) Functions

sin , cos , tan , cosec, sec , cot

and inverses

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rg

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Circular Functions - Phase Shifts

	sin	cos	tan
$\theta + 90°$	$^{+}\cos\theta$	$^{-}\sin\theta$	$^{-}\cot\theta$
$\theta - 90°$	$^{-}\cos\theta$	$^{+}\sin\theta$	$^{-}\cot\theta$
$90° - \theta$	$^{+}\cos\theta$	$^{+}\sin\theta$	$^{+}\cot\theta$
$\theta + 180°$	$^{-}\sin\theta$	$^{-}\cos\theta$	$^{+}\tan\theta$
$\theta - 180°$	$^{-}\sin\theta$	$^{-}\cos\theta$	$^{+}\tan\theta$
$180° - \theta$	$^{+}\sin\theta$	$^{-}\cos\theta$	$^{-}\tan\theta$
$\theta + 270°$	$^{-}\cos\theta$	$^{+}\sin\theta$	$^{-}\cot\theta$
$\theta - 270°$	$^{+}\cos\theta$	$^{-}\sin\theta$	$^{-}\cot\theta$
$270° - \theta$	$^{-}\cos\theta$	$^{-}\sin\theta$	$^{+}\cot\theta$
$\theta + 360°$	$^{+}\sin\theta$	$^{+}\cos\theta$	$^{+}\tan\theta$
$\theta - 360°$	$^{+}\sin\theta$	$^{+}\cos\theta$	$^{+}\tan\theta$
$360° - \theta$	$^{-}\sin\theta$	$^{+}\cos\theta$	$^{-}\tan\theta$
$^{-}\theta$	$^{-}\sin\theta$	$^{+}\cos\theta$	$^{-}\tan\theta$
$^{-}\theta$	$^{-}\mathrm{cosec}\,\theta$	$^{+}\sec\theta$	$^{-}\cot\theta$
functions are	odd	even	odd

Hyperbolic Functions - Sign Changes

^{-}x	$^{-}\sinh x$	$^{+}\cosh x$	$^{-}\tanh x$
^{-}x	$^{-}\mathrm{cosech}\, x$	$^{+}\mathrm{sech}\, x$	$^{-}\coth x$
functions are	odd	even	odd

Notes

The sign changes for cosine *and* sine *cover all the 8 permutations of* + *and* –

cosines *are even functions* ($\cos\theta = \cos{}^{-}\theta$) , sines *are odd functions* ($\sin\theta = {}^{-}\sin{}^{-}\theta$)

Circular Function Values

		sin		cos		tan	
$-\pi$	$-180°$	0	$-\sqrt{0}/2$	-1	$-\sqrt{4}/2$	0	$\sqrt{0}/\sqrt{4}$
$-5\pi/6$	$-150°$	$-1/2$	$-\sqrt{1}/2$	$-\sqrt{3}/2$	$-\sqrt{3}/2$	$\sqrt{3}/3$	$\sqrt{1}/\sqrt{3}$
$-3\pi/4$	$-135°$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	1	$\sqrt{2}/\sqrt{2}$
$-2\pi/3$	$-120°$	$-\sqrt{3}/2$	$-\sqrt{3}/2$	$-1/2$	$-\sqrt{1}/2$	$\sqrt{3}$	$\sqrt{3}/\sqrt{1}$
$-\pi/2$	$-90°$	-1	$-\sqrt{4}/2$	0	$\sqrt{0}/2$	$\mp\infty$	$-\sqrt{4}/\sqrt{0}$
$-\pi/3$	$-60°$	$-\sqrt{3}/2$	$-\sqrt{3}/2$	$1/2$	$\sqrt{1}/2$	$-\sqrt{3}$	$-\sqrt{3}/\sqrt{1}$
$-\pi/4$	$-45°$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	$\sqrt{2}/2$	$\sqrt{2}/2$	-1	$-\sqrt{2}/\sqrt{2}$
$-\pi/6$	$-30°$	$-1/2$	$-\sqrt{1}/2$	$\sqrt{3}/2$	$\sqrt{3}/2$	$-\sqrt{3}/3$	$-\sqrt{1}/\sqrt{3}$
0	$0°$	0	$\sqrt{0}/2$	1	$\sqrt{4}/2$	0	$\sqrt{0}/\sqrt{4}$
$\pi/12$	$15°$	$\frac{1}{4}(\sqrt{6}-\sqrt{2})$		$\frac{1}{4}(\sqrt{6}+\sqrt{2})$		$2-\sqrt{3}$	
$\pi/6$	$30°$	$1/2$	$\sqrt{1}/2$	$\sqrt{3}/2$	$\sqrt{3}/2$	$\sqrt{3}/3$	$\sqrt{1}/\sqrt{3}$
$\pi/5$	$36°$	$\frac{1}{2}\sqrt{\{\frac{1}{2}(5-\sqrt{5})\}}$		$\frac{1}{4}(1+\sqrt{5})$			
$\pi/4$	$45°$	$\sqrt{2}/2$	$\sqrt{2}/2$	$\sqrt{2}/2$	$\sqrt{2}/2$	1	$\sqrt{2}/\sqrt{2}$
$3\pi/10$	$54°$	$\frac{1}{4}(1+\sqrt{5})$		$\frac{1}{2}\sqrt{\{\frac{1}{2}(5-\sqrt{5})\}}$			
$\pi/3$	$60°$	$\sqrt{3}/2$	$\sqrt{3}/2$	$1/2$	$\sqrt{1}/2$	$\sqrt{3}$	$\sqrt{3}/\sqrt{1}$
$5\pi/12$	$75°$	$\frac{1}{4}(\sqrt{6}+\sqrt{2})$		$\frac{1}{4}(\sqrt{6}-\sqrt{2})$		$2+\sqrt{3}$	
$\pi/2$	$90°$	1	$\sqrt{4}/2$	0	$\sqrt{0}/2$	$\pm\infty$	$\sqrt{4}/\sqrt{0}$
$2\pi/3$	$120°$	$\sqrt{3}/2$	$\sqrt{3}/2$	$-1/2$	$-\sqrt{1}/2$	$-\sqrt{3}$	$-\sqrt{3}/\sqrt{1}$
$3\pi/4$	$135°$	$\sqrt{2}/2$	$\sqrt{2}/2$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	-1	$-\sqrt{2}/\sqrt{2}$
$5\pi/6$	$150°$	$1/2$	$\sqrt{1}/2$	$-\sqrt{3}/2$	$-\sqrt{3}/2$	$-\sqrt{3}/3$	$-\sqrt{1}/\sqrt{3}$
π	$180°$	0	$-\sqrt{0}/2$	-1	$-\sqrt{4}/2$	0	$\sqrt{0}/\sqrt{4}$
$7\pi/6$	$210°$	$-1/2$	$-\sqrt{1}/2$	$-\sqrt{3}/2$	$-\sqrt{3}/2$	$\sqrt{3}/3$	$\sqrt{1}/\sqrt{3}$
$5\pi/4$	$225°$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	1	$\sqrt{2}/\sqrt{2}$
$4\pi/3$	$240°$	$-\sqrt{3}/2$	$-\sqrt{3}/2$	$-1/2$	$-\sqrt{1}/2$	$\sqrt{3}$	$\sqrt{3}/\sqrt{1}$
$3\pi/2$	$270°$	-1	$-\sqrt{4}/2$	0	$\sqrt{0}/2$	$\mp\infty$	$-\sqrt{4}/\sqrt{0}$
$5\pi/3$	$300°$	$-\sqrt{3}/2$	$-\sqrt{3}/2$	$1/2$	$\sqrt{1}/2$	$-\sqrt{3}$	$-\sqrt{3}/\sqrt{1}$
$7\pi/4$	$315°$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	$\sqrt{2}/2$	$\sqrt{2}/2$	-1	$-\sqrt{2}/\sqrt{2}$
$11\pi/6$	$330°$	$-1/2$	$-\sqrt{1}/2$	$\sqrt{3}/2$	$\sqrt{3}/2$	$-\sqrt{3}/3$	$-\sqrt{1}/\sqrt{3}$
2π	$360°$	0	$-\sqrt{0}/2$	1	$\sqrt{4}/2$	0	$-\sqrt{0}/\sqrt{4}$

Basic Trigonometric (Circular) Relationships

$$1 = \cos^2\theta + \sin^2\theta$$

$$\sec^2\theta = 1 + \tan^2\theta \qquad \operatorname{cosec}^2\theta = 1 + \cot^2\theta$$

Addition Formulae

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$$

$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$

Double Angle Formulae [1]

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

Half Angle Formulae

$$\sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta)$$

$$\cos^2\theta = \tfrac{1}{2}(1 + \cos 2\theta)$$

$$\tan^2\theta = {}^{1-\cos 2\theta}/_{1+\cos 2\theta}$$

$$\cos 2\theta = {}^{1-\tan^2\theta}/_{1+\tan^2\theta}$$

Product Formulae [2]

$$\sin\theta\cos\phi = \tfrac{1}{2}[\sin(\theta+\phi) + \sin(\theta-\phi)]$$

$$\cos\theta\sin\phi = \tfrac{1}{2}[\sin(\theta+\phi) - \sin(\theta-\phi)]$$ *because* $\sin{}^-\theta = {}^-\sin\theta$

$$\cos\theta\cos\phi = \tfrac{1}{2}[\cos(\theta+\phi) + \cos(\theta-\phi)]$$

$$\sin\theta\sin\phi = -\tfrac{1}{2}[\cos(\theta+\phi) - \cos(\theta-\phi)]$$

Factor / Sum Formulae

set $\theta + \phi = P$ *set* $\theta - \phi = Q$

then $\theta = \tfrac{1}{2}(P + Q)$ *then* $\phi = \tfrac{1}{2}(P - Q)$

$$\sin P + \sin Q = 2\sin[\tfrac{1}{2}(P+Q)]\cos[\tfrac{1}{2}(P-Q)]$$ [3]

$$\sin P - \sin Q = 2\cos[\tfrac{1}{2}(P+Q)]\sin[\tfrac{1}{2}(P-Q)]$$

$$\cos P + \cos Q = 2\cos[\tfrac{1}{2}(P+Q)]\cos[\tfrac{1}{2}(P-Q)]$$

$$\cos P - \cos Q = {}^-2\sin[\tfrac{1}{2}(P+Q)]\sin[\tfrac{1}{2}(P-Q)]$$

Notes

[1] *For further insight into asymmetry between* sine *and* cosine *see section on power series*

[2] *These are useful for integration but were originally derived by Vièta as a precusor to logs.*

[3] *This form gives the resultant addition of two frequencies*

Trigonometrical (Circular) Relationships 2

define $\tan\theta = {}^{\sin\theta}/_{\cos\theta}$ $\qquad \tan(\tfrac{1}{2}\pi - \phi) = \cot\theta$

$\tan(\theta + \phi) = {}^{\tan\theta + \tan\phi}/_{1 - \tan\theta\tan\phi}$ $\qquad$ *try setting* $\theta = \tfrac{1}{4}\pi$

$\tan(\theta - \phi) = {}^{\tan\theta - \tan\phi}/_{1 + \tan\theta\tan\phi}$

$\tan 2\theta = {}^{2\tan\theta}/_{1 - \tan^2\theta}$

$\sin 2\theta = {}^{2\tan\theta}/_{1 + \tan^2\theta}$ $\qquad$ *from* $\sec^2\theta = 1 + \tan^2\theta$

$\cos 2\theta = {}^{1 - \tan^2\theta}/_{1 + \tan^2\theta}$

$\cot(\theta + \phi) = {}^{\cot\phi \,.\, \cot\theta - 1}/_{\cot\phi + \cot\theta}$

$\cot(\theta - \phi) = {}^{\cot\phi\cot\theta + 1}/_{\cot\phi - \cot\theta}$

$\cot 2\theta = {}^{\cot^2\theta - 1}/_{2\cot\theta}$

$\cot\theta = \mathrm{cosec}\,2\theta + \cot 2\theta$

$\tan\theta = \mathrm{cosec}\,2\theta - \cot 2\theta$ $\qquad$ *cf* $\int \mathrm{cosec}\,\theta\, d\theta$

$\cot(\theta - \tfrac{1}{2}\pi) = \cot 2\theta - \mathrm{cosec}\,2\theta = {}^{-}\tan\theta$

$\tan\theta = {}^{1}/_{\mathrm{cosec}\,2\theta + \cot 2\theta}$

$\cot\theta = {}^{1}/_{\mathrm{cosec}\,2\theta - \cot 2\theta}$

$\tan(\theta - \tfrac{1}{2}\pi) = {}^{1}/_{\cot 2\theta - \mathrm{cosec}\,2\theta} = {}^{-}\cot\theta$

$\tan\theta = {}^{\sin 2\theta}/_{1 + \cos 2\theta}$

$\cot\theta = {}^{\sin 2\theta}/_{1 - \cos 2\theta}$

$\tan(\theta - \tfrac{1}{2}\pi) = {}^{\sin 2\theta}/_{\cos 2\theta - 1} = {}^{-}\cot\theta$

$\cot(\theta + \tfrac{1}{4}\pi) = {}^{\cos 2\theta}/_{1 + \sin 2\theta}$

$\tan(\theta + \tfrac{1}{4}\pi) = {}^{\cos 2\theta}/_{1 - \sin 2\theta}$

$\cot(\theta - \tfrac{1}{4}\pi) = {}^{\cos 2\theta}/_{\sin 2\theta - 1}$

$a\sin\theta \pm b\cos\theta = R\sin(\theta \pm \phi)$ $\qquad 0 < \alpha < 90^\circ$ or $(\tfrac{1}{2}\pi)$

$a\cos\theta \pm b\sin\theta = R\cos(\theta \mp \phi)$

where $R\cos\phi = a$ *and* $R\sin\phi = b$

$R = \sqrt{(a^2 + b^2)}$

and $\phi = \tan^{-1} {}^{b}/_{a}$

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Trigonometrical (Circular) Relationships 3

$$\frac{1 + \sin\theta}{1 - \sin\theta} = (\sec\theta + \tan\theta)^2 = \tan^2\left(\tfrac{1}{2}\theta + \tfrac{1}{4}\pi\right)$$

$$\frac{1 - \sin\theta}{1 + \sin\theta} = (\sec\theta - \tan\theta)^2 = \cot^2\left(\tfrac{1}{2}\theta + \tfrac{1}{4}\pi\right)$$

$$\frac{\sin\theta + 1}{\sin\theta - 1} = -(\sec\theta + \tan\theta)^2 = \cot^2\left(\tfrac{1}{2}\theta - \tfrac{1}{4}\pi\right)$$

$$\frac{\sin\theta - 1}{\sin\theta + 1} = -(\sec\theta - \tan\theta)^2 = \tan^2\left(\tfrac{1}{2}\theta - \tfrac{1}{4}\pi\right)$$

$$\frac{1 + \cos\theta}{1 - \cos\theta} = (\operatorname{cosec}\theta + \cot\theta)^2 = \cot^2\left(\tfrac{1}{2}\theta\right)$$

$$\frac{1 - \cos\theta}{1 + \cos\theta} = (\operatorname{cosec}\theta - \cot\theta)^2 = \tan^2\left(\tfrac{1}{2}\theta\right)$$

$$\frac{\cos\theta + 1}{\cos\theta - 1} = -(\operatorname{cosec}\theta + \cot\theta)^2 = \tan^2\left(\tfrac{1}{2}\theta - \tfrac{1}{2}\pi\right)$$

$$\frac{\cos\theta - 1}{\cos\theta + 1} = -(\operatorname{cosec}\theta - \cot\theta)^2 = \cot^2\left(\tfrac{1}{2}\theta - \tfrac{1}{2}\pi\right)$$

$$\frac{1 + \tan\theta}{1 - \tan\theta} = \sec 2\theta + \tan 2\theta = \tan\left(\theta + \tfrac{1}{4}\pi\right)$$

$$\frac{1 - \tan\theta}{1 + \tan\theta} = \sec 2\theta - \tan 2\theta = \cot\left(\theta + \tfrac{1}{4}\pi\right)$$

$$\frac{\tan\theta + 1}{\tan\theta - 1} = -(\sec 2\theta + \tan 2\theta) = \cot\left(\theta - \tfrac{1}{4}\pi\right)$$

$$\frac{\tan\theta - 1}{\tan\theta + 1} = -(\sec 2\theta - \tan 2\theta) = \tan\left(\theta - \tfrac{1}{4}\pi\right)$$

Inverse Circular Functions 1

$\sin^{-1}x$	=	$-i \ln\left(ix \pm \sqrt{1 - x^2}\right)$	$-1 \le x \le +1$
$\sin^{-1}x$	=	$-i \ln\left(ix \pm i\sqrt{x^2 - 1}\right)$	$-1 \le x \le +1$
$\operatorname{cosec}^{-1}x$	=	$-i \ln\left(\frac{i}{x} \pm \sqrt{1 - \frac{1}{x^2}}\right)$	$x \le -1$ *and* $+1 \le x$
$\cos^{-1}x$	=	$-i \ln\left(x \pm \sqrt{x^2 - 1}\right)$	$-1 \le x \le +1$
$\cos^{-1}x$	=	$-i \ln\left(x \pm i\sqrt{1 - x^2}\right)$	$-1 \le x \le +1$
$\sec^{-1}x$	=	$-i \ln\left(\frac{1}{x} \pm \sqrt{\frac{1}{x^2} - 1}\right)$	$x \le -1$ *and* $+1 \le x$
$\tan^{-1}x$	=	$\tfrac{1}{2} i \ln\left(\frac{i + x}{i - x}\right)$	$x \in P$
$\cot^{-1}x$	=	$\tfrac{1}{2} i \ln\left(\frac{ix + 1}{ix - 1}\right)$	$x \in P$

Notes

Inverse circular functions used to be termed "goniometric". They are many-valued.

Inverse Circular Functions 2

if $\sin b$	$=$	$^1/_a$	*then* $\operatorname{cosec} b$	$=$	a
hence b	$=$	$\sin^{-1}(^1/_a)$	*and also* b	$=$	$\operatorname{cosec}^{-1} a$
hence $\operatorname{cosec}^{-1}(a)$	$=$	$\sin^{-1}(^1/_a)$			

$\sin^{-1}x$ *is an odd function*

$\sin^{-1}(^-x)$	$=$	$^-\sin^{-1}x$	$^-1 \le x \le {}^+1$
$\operatorname{cosec}^{-1}x$	$=$	$\sin^{-1}(^1/_x)$	$x \le {}^-1$ *and* $^+1 \le x$

$(\cos^{-1}x - {}^{\pi}/_2)$ *is an odd function*

$\cos^{-1}(^-x)$	$=$	$^-\cos^{-1}x + \pi$	$^-1 \le x \le {}^+1$
$\sec^{-1}x$	$=$	$\cos^{-1}(^1/_x)$	$x \le {}^-1$ *and* $^+1 \le x$

$\tan^{-1}x$ *is an odd function*

$\tan^{-1}(^-x)$	$=$	$^-\tan^{-1}x$	$x \in P$
$\cot^{-1}x$	$=$	$\tan^{-1}(^1/_x)$	$x \in P$

Inverse Circular Functions 3

$\sin^{-1}x + \cos^{-1}x$	$=$	$\frac{1}{2}\pi$	$^-1 \le x \le {}^+1$
$\operatorname{cosec}^{-1}x + \sec^{-1}x$	$=$	$\frac{1}{2}\pi$	$x \le {}^-1$ *and* $^+1 \le x$
$\tan^{-1}x + \cot^{-1}x$	$=$	$^-\frac{1}{2}\pi$	$x < 0$ †
$\tan^{-1}x + \cot^{-1}x$	$=$	$\frac{1}{2}\pi$	$x > 0$ †
$\operatorname{cosec}^{-1}(^-x)$	$=$	$^-\operatorname{cosec}^{-1}x$	$x \le {}^-1$ *and* $^+1 \le x$
$\sec^{-1}(^-x)$	$=$	$\pi - \sec^{-1}x$	$x \le {}^-1$ *and* $^+1 \le x$
$\cot^{-1}(^-x)$	$=$	$^-\cot^{-1}x$	$x \in P \quad x \ne 0$

Notes

$\tan^{-1}x \pm \tan^{-1}y = \tan^{-1}\{(x \pm y) / (1 \mp xy)\}$

Inverse circular functions give the arc length of a sector on the unit circle $x^2 + y^2 = 1$

This is why the functions may also be written arcsin, arccos, arctan *etc.*

But note it's arsinh, arcosh, artanh *etc. because this function relates to area*

Trigonometric (Circular) Power Series Basic Relationship

$z^n + {}^1/z^n = 2\cos n\theta$ $\qquad$ $z^n - {}^1/z^n = 2i \sin n\theta$ $\quad (z^n = \text{cis}\, n\theta)$

even powers $\cos^n$ in cosines

Expand $(z + {}^1/_z)^n$ *for n even*

$\cos^2\theta = {}^{+1}/_{2^1} (1\cos 2\theta + 1)$

$\cos^4\theta = {}^{+1}/_{2^3} (1\cos 4\theta + 4\cos 2\theta + 3)$

$\cos^6\theta = {}^{+1}/_{2^5} (1\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$

$\cos^8\theta = {}^{+1}/_{2^7} (1\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35)$

odd powers $\cos^n$ in cosines

Expand $(z + {}^1/_z)^n$ *for n odd*

$\cos^3\theta = {}^{+1}/_{2^2} (1\cos 3\theta + 3\cos \theta)$

$\cos^5\theta = {}^{+1}/_{2^4} (1\cos 5\theta + 5\cos 3\theta + 10\cos \theta)$

$\cos^7\theta = {}^{+1}/_{2^6} (1\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos \theta)$

even powers $\sin^n$ in cosines

Expand $(z - {}^1/_z)^n$ *for n even*

$\sin^2\theta = {}^{+1}/_{2^1} (1 - 1\cos 2\theta)$

$\sin^4\theta = {}^{+1}/_{2^3} (3 - 4\cos 2\theta + 1\cos 4\theta)$

$\sin^6\theta = {}^{+1}/_{2^5} (10 - 15\cos 2\theta + 6\cos 4\theta - 1\cos 6\theta)$

$\sin^8\theta = {}^{+1}/_{2^7} (35 - 56\cos 2\theta + 28\cos 4\theta - 8\cos 6\theta + 1\cos 8\theta)$

odd powers $\sin^n$ in sines

Expand $(z - {}^1/_z)^n$ *for n odd*

$\sin^3\theta = {}^{-1}/_{2^2} (\sin 3\theta - 3\sin \theta)$

$\sin^5\theta = {}^{+1}/_{2^4} (\sin 5\theta - 5\sin 3\theta + 10\sin \theta)$

$\sin^7\theta = {}^{-1}/_{2^6} (\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin \theta)$

Notes

There is no apparent way to produce the following series expansions

even powers $\cos^n$ in sines $\qquad$ **odd powers $\cos^n$ in sines**

even powers $\sin^n$ in sines $\qquad$ **odd powers $\sin^n$ in cosines**

and this shows why 3 of the 4 double angle formulae are cos *and only one for* sin

Expansion cos(nθ) and sin(nθ) in power series of cos θ and sin θ

from de Moivres theorem $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$ *we can create power series by expanding* $(\cos\theta + i\sin\theta)$ *and equating real and imaginary terms.*

expansion cos(nθ) for n odd

Coefficients are Chebyshev Polynomials $T_n(x)$

$$\cos\theta = 1\cos\theta$$

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

$$\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$$

expansion cos(nθ) for n even

Coefficients are Chebyshev Polynomials $T_n(x)$

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$$

$$\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$$

$$\cos 8\theta = 128\cos^8\theta - 256\cos^6\theta + 160\cos^4\theta - 32\cos^2\theta + 1$$

expansion sin(nθ) for n odd

nb cf coefficients with cos expansions.

$$\sin\theta = 1\sin\theta$$

$$\sin 3\theta = {}^{-}4\sin^3\theta + 3\sin\theta$$

$$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$$

$$\sin 7\theta = {}^{-}64\sin^7\theta + 112\sin^5\theta - 56\sin^3\theta + 7\sin\theta$$

expansion sin(nθ) for n even

nb annoying $(\sin\theta\cos\theta)$ *term recurs.*

$$\sin 2\theta = 2\sin\theta\cos\theta\,(1)$$

$$\sin 4\theta = 2\sin\theta\cos\theta\,({}^{-}4\sin^2\theta + 2)$$

$$\sin 6\theta = 2\sin\theta\cos\theta\,(16\sin^4\theta - 16\sin^2\theta + 3)$$

$$\sin 8\theta = 2\sin\theta\cos\theta\,({}^{-}64\sin^6\theta + 96\sin^4\theta - 40\sin^2\theta + 4)$$

Notes

$T_n(x)$ *is generated by* $T_0(x) = 1$; $T_1(x) = x$ *and thereafter* $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$

Further Circular Function Values

		sin		cos	
0	$0°$	0		1	
$^{\pi}/_{10}$	$18°$	$\frac{1}{4}(\sqrt{5}-1)$	$\frac{1}{2}(\phi-1)$	$\frac{1}{4}\sqrt{(10+2\sqrt{5})}$	$\phi(\sqrt{(1-\frac{1}{4}\phi^2)}$
$^{\pi}/_{5}$	$36°$	$\frac{1}{2}\sqrt{(\frac{1}{2}(5-\sqrt{5}))}$	$(\sqrt{(1-\frac{1}{4}\phi^2)}$	$\frac{1}{4}(\sqrt{5}+1)$	$\frac{1}{2}(\phi)$
$^{3\pi}/_{10}$	$54°$	$\frac{1}{4}(\sqrt{5}+1)$	$\frac{1}{2}(\phi)$	$\frac{1}{2}\sqrt{(\frac{1}{2}(5-\sqrt{5}))}$	$(\sqrt{(1-\frac{1}{4}\phi^2)}$
$^{2\pi}/_{5}$	$72°$	$\frac{1}{4}\sqrt{(10+2\sqrt{5})}$	$\phi(\sqrt{(1-\frac{1}{4}\phi^2)}$	$\frac{1}{4}(\sqrt{5}-1)$	$\frac{1}{2}(\phi-1)$
$^{\pi}/_{2}$	$90°$	1		0	
$^{3\pi}/_{5}$	$108°$	$\frac{1}{4}\sqrt{(10+2\sqrt{5})}$	$\phi(\sqrt{(1-\frac{1}{4}\phi^2)}$	$^{-}\frac{1}{4}(\sqrt{5}-1)$	$^{-}\frac{1}{2}(\phi-1)$
$^{7\pi}/_{10}$	$126°$	$\frac{1}{4}(\sqrt{5}+1)$	$\frac{1}{2}(\phi)$	$^{-}\frac{1}{2}\sqrt{(\frac{1}{2}(5-\sqrt{5}))}$	$^{-}(\sqrt{(1-\frac{1}{4}\phi^2)}$
$^{4\pi}/_{5}$	$144°$	$\frac{1}{2}\sqrt{(\frac{1}{2}(5-\sqrt{5}))}$	$(\sqrt{(1-\frac{1}{4}\phi^2)}$	$^{-}\frac{1}{4}(\sqrt{5}+1)$	$^{-}\frac{1}{2}(\phi)$
$^{9\pi}/_{10}$	$162°$	$\frac{1}{4}(\sqrt{5}-1)$	$\frac{1}{2}(\phi-1)$	$^{-}\frac{1}{4}\sqrt{(10+2\sqrt{5})}$	$^{-}\phi(\sqrt{(1-\frac{1}{4}\phi^2)}$
π	$180°$	0		$^{-}1$	

Notes on Fibonacci Series

The golden ratio $\phi = \frac{1}{2}(1+\sqrt{5})$.

ϕ *is the limit of the ratios of adjacent terms of the Fibonacci series*

$\phi^a - (1-\phi^a) / \sqrt{5}$ *gives the* ath *Fibonacci number.*

The first 15 *Fibonacci numbers are* 0, 1, 1, 2, ,3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377

Remember every sine *and* cosine *value are related by the equation* $\sin^2 + \cos^2 = 1$

Demonstrating this algebraically at values such as $^{2\pi}/_{5}$ *is actually a bit tricky.*

The insight into all this lies in examining the 5th *roots of* 1 *using complex numbers*

Further Notes on Expansion of cos nθ and sin nθ

Expansion of cos nθ

$$\cos n\theta = \cos^n\theta - \{^{n(n-1)}/_{2!}\} \cos^{n-2}\theta \sin^2\theta + \{^{n(n-1)(n-2)(n-3)}/_{4!}\} \cos^{n-4}\theta \sin^4\theta \ldots$$

Expansion of sin nθ

$$\sin n\theta = n\cos^{n-1}\theta\sin\theta - \{^{n(n-1)(n-2)}/_{3!}\}\cos^{n-3}\theta\sin^3\theta + \{^{n(n-1)(n-2)(n-3)(n-4)}/_{5!}\}\cos^{n-5}\theta\sin^5\theta \ldots$$

Each expansion continues until one term of n(n–1)(n–2)… *equates to zero*

Circular "Functions of Functions" Mapping

$\cos(\cos^{-1}x)$	=	x	$\sec(\cos^{-1})$	=	$^1/_x$
$\sec(\sec^{-1}x)$	=	x	$\cos(\sec^{-1}x)$	=	$^1/_x$
$\sin(\sin^{-1}x)$	=	x	$\text{cosec}(\sin^{-1}x)$	=	$^1/_x$
$\text{cosec}(\text{cosec}^{-1}x)$	=	x	$\sin(\text{cosec}^{-1}x)$	=	$^1/_x$
$\tan(\tan^{-1}x)$	=	x	$\cot(\tan^{-1}x)$	=	$^1/_x$
$\cot(\cot^{-1}x)$	=	x	$\tan(\cot^{-1}x)$	=	$^1/_x$
$\sec(\tan^{-1}x)$	=	$\sqrt{(x^2 + 1)}$	$\cos(\tan^{-1}x)$	=	$^1/_{\sqrt{(x^2+1)}}$
$\text{cosec}(\cot^{-1}x)$	=	$\sqrt{(x^2 + 1)}$	$\sin(\cot^{-1}x)$	=	$^1/_{\sqrt{(x^2+1)}}$
$\sec(\cot^{-1})$	=	$^{\sqrt{(x^2+1)}}/_x$	$\cos(\cot^{-1})$	=	$^x/_{\sqrt{(x^2+1)}}$
$\text{cosec}(\tan^{-1}x)$	=	$^{\sqrt{(x^2+1)}}/_x$	$\sin(\tan^{-1}x)$	=	$^x/_{\sqrt{(x^2+1)}}$
$\tan(\sec^{-1}x)$	=	$\sqrt{(x^2 - 1)}$	$\cot(\sec^{-1}x)$	=	$^1/_{\sqrt{(x^2-1)}}$
$\cot(\text{cosec}^{-1}x)$	=	$\sqrt{(x^2 - 1)}$	$\tan(\text{cosec}^{-1}x)$	=	$^1/_{\sqrt{(x^2-1)}}$
$\cos(\text{cosec}^{-1}x)$	=	$^{\sqrt{(x^2-1)}}/_x$	$\sec(\text{cosec}^{-1}x)$	=	$^x/_{\sqrt{(x^2-1)}}$
$\sin(\sec^{-1}x)$	=	$^{\sqrt{(x^2-1)}}/_x$	$\text{cosec}(\sec^{-1}x)$	=	$^x/_{\sqrt{(x^2-1)}}$
$\cos(\sin^{-1}x)$	=	$\sqrt{(1 - x^2)}$	$\sec(\sin^{-1}x)$	=	$^1/_{\sqrt{(1-x^2)}}$
$\sin(\cos^{-1}x)$	=	$\sqrt{(1 - x^2)}$	$\text{cosec}(\cos^{-1}x)$	=	$^1/_{\sqrt{(1-x^2)}}$
$\tan(\cos^{-1}x)$	=	$^{\sqrt{(1-x^2)}}/_x$	$\cot(\cos^{-1}x)$	=	$^x/_{\sqrt{(1-x^2)}}$
$\cot(\sin^{-1}x)$	=	$^{\sqrt{(1-x^2)}}/_x$	$\tan(\sin^{-1}x)$	=	$^x/_{\sqrt{(1-x^2)}}$

Notes

Taking the functions sin cos tan *and* cosec sec cot *we can perm into 36 relationships*

If we arrange these by the result rather than the relationship the following pattern emerges

Personally I got a great deal of satisfaction in structuring this which took some time!

Functions of Functions

$\sin(\cos^{-1} {}^{x}/_{a})$	$= \sqrt{(a^2 - x^2)}/_{\lvert a \rvert}$	$\sin(\sec^{-1} {}^{x}/_{a})$	$= \sqrt{(x^2 - a^2)}/_{\lvert x \rvert}$
$\sin(\tan^{-1} {}^{x}/_{a})$	$= x\,\mathrm{sign}(a)/_{\sqrt{(x^2 + a^2)}}$	$\sin(\cot^{-1} {}^{x}/_{a})$	$= \lvert a \rvert/_{\sqrt{(x^2 + a^2)}}$
$\cos(\sin^{-1} {}^{x}/_{a})$	$= \sqrt{(a^2 - x^2)}/_{\lvert a \rvert}$	$\cos(\mathrm{cosec}^{-1} {}^{x}/_{a})$	$= \sqrt{(x^2 - a^2)}/_{\lvert x \rvert}$
$\cos(\tan^{-1} {}^{x}/_{a})$	$= \lvert a \rvert/_{\sqrt{(x^2 + a^2)}}$	$\cos(\cot^{-1} {}^{x}/_{a})$	$= x\,\mathrm{sign}(a)/_{\sqrt{(x^2 + a^2)}}$
$\tan(\sin^{-1} {}^{x}/_{a})$	$= x\,\mathrm{sign}(a)/_{\sqrt{(a^2 - x^2)}}$	$\tan(\mathrm{cosec}^{-1} {}^{x}/_{a})$	$= a\,\mathrm{sign}(x)/_{\sqrt{(x^2 - a^2)}}$
$\tan(\cos^{-1} {}^{x}/_{a})$	$= \sqrt{(a^2 - x^2)}\,\mathrm{sign}(a)/_{x}$	$\tan(\sec^{-1} {}^{x}/_{a})$	$= \sqrt{(x^2 - a^2)}\,\mathrm{sign}(x)/_{a}$

For functions cosec, sec *and* cot *just invert the corresponding function* sin, cos *and* tan.

For functions cosec^{-1}, $\sec^{-1}$ *and* $\cot^{-1}$, *coeff.* **x** *and* **a** *change places. (1 exception)*

Further Identities

$$(\sin\theta + \cos\theta)^2 = 1 + \sin 2\theta$$

$$(\sin\theta - \cos\theta)^2 = 1 - \sin 2\theta$$

$$\cos(\theta + \phi)\cos(\theta - \phi) = \cos^2\theta - \sin^2\phi$$

$$\sin(\theta + \phi)\sin(\theta - \phi) = \cos^2\phi - \cos^2\theta$$

From $\tan 2\theta = {}^{2\tan\theta}/_{1 - \tan^2\theta}$ *we can derive the following relations*

$$\tan \tfrac{1}{2}\theta = t \qquad\qquad \sin\theta = {}^{2t}/_{1 + t^2}$$

$$^{d\theta}/_{dt} = {}^{2}/_{1 + t^2} \qquad\qquad \cos\theta = {}^{1 - t^2}/_{1 + t^2}$$

$$\tan\theta = {}^{2t}/_{1 - t^2}$$

General Solutions

for $\sin x = \sin\alpha$	*then*	x	$= n\pi + (-1)^n\alpha$
for $\cos x = \cos\alpha$	*then*	x	$= 2n\pi \pm \alpha$
for $\tan x = \tan\alpha$	*then*	x	$= n\pi + \alpha$

Domain and Range of Trigonometric Functions

Function	Domain	Range	Type	Exceptions
$\sin\theta$	$\theta \in P$	$^{-}1 \le y \le {}^{+}1$	*odd function*	
$\cos\theta$	$\theta \in P$	$^{-}1 \le y \le {}^{+}1$	*even function*	
$\tan\theta$	$\theta \in P$	$y \in P$	*odd function*	$x \ne \pm \frac{1}{2}\pi \pm 2n\pi$
$\mathrm{cosec}\,\theta$	$\theta \in P$	$y \le {}^{-}1 \quad {}^{+}1 \le y$	*odd function*	$x \ne \pm \pi \pm 2n\pi$
$\sec\theta$	$\theta \in P$	$y \le {}^{-}1 \quad {}^{+}1 \le y$	*even function*	$x \ne \pm \frac{1}{2}\pi \pm 2n\pi$
$\cot\theta$	$\theta \in P$	$y \in P$	*odd function*	$x \ne \pm \pi \pm 2n\pi$
$\sin^{-1}x$	$^{-}1 \le x \le {}^{+}1$	$^{-}\frac{1}{2}\pi \le y \le {}^{+}\frac{1}{2}\pi$	*odd function*	
$\cos^{-1}x$	$^{-}1 \le x \le {}^{+}1$	$0 \le y \le {}^{+}\pi$	*nb* $(\cos^{-1}x - \pi)$ *is an odd function*	
$\tan^{-1}x$	$x \in P$	$^{-}\frac{1}{2}\pi < y < {}^{+}\frac{1}{2}\pi$	*odd function*	
$\mathrm{cosec}^{-1}x$	$x \le {}^{-}1 \quad {}^{+}1 \le x$	$^{-}\frac{1}{2}\pi \le y \le {}^{+}\frac{1}{2}\pi$	*odd function*	
$\sec^{-1}x$	$x \le {}^{-}1 \quad {}^{+}1 \le x$	$0 \le y \le {}^{+}\pi$	*nb* $(\sec^{-1}x - \pi)$ *is an odd function*	
$\cot^{-1}x$	$x \in P$	$^{-}\frac{1}{2}\pi < y < {}^{+}\frac{1}{2}\pi$	*odd function*	$x \ne 0$
$\sinh x$	$x \in P$	$y \in P$	*odd function*	
$\cosh x$	$x \in P$	$^{+}1 \le y$	*even function*	*A catenary curve*
$\tanh x$	$x \in P$	$^{-}1 < y < {}^{+}1$	*odd function*	
$\mathrm{cosech}\,x$	$x \in P$	$y \in P$	*odd function*	$x \ne 0 \quad y \ne 0$
$\mathrm{sech}\,x$	$x \in P$	$0 < y \le {}^{+}1$	*even function* †	*max. at* $x = 0$
$\coth x$	$x \in P$	$y < {}^{-}1 \quad {}^{+}1 < y$	*odd function*	$x \ne 0$
$\sinh^{-1}x$	$x \in P$	$y \in P$	*odd function*	
$\cosh^{-1}x$	$1 < x$	$0 \le y$		
$\tanh^{-1}x$	$^{-}1 < x < {}^{+}1$	$^{-}\pi < y < {}^{+}\pi$	*odd function*	
$\mathrm{cosech}^{-1}x$	$x \in P$	$y \in P$	*odd function*	$x \ne 0$
$\mathrm{sech}^{-1}x$	$0 < x \le {}^{+}1$	$0 \le y < {}^{+}\pi$		
$\coth^{-1}x$	$x < {}^{-}1 \quad {}^{+}1 < x$	$y \in P$	*odd function*	

Notes

† sech^2 *is a bell shaped distribution occurring in the natural world.*

First Order Chebyshev Polynomials

$T_0(x) = 1$

$T_1(x) = x$

$T_2(x) = 2x^2 - 1$

$T_3(x) = 4x^3 - 3x$

$T_4(x) = 8x^4 - 8x^2 + 1$

$T_5(x) = 16x^5 - 20x^3 + 5x$

$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$

$T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$

$T_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$

$T_0(x) = 1$

$T_1(x) = x$

$T_{n+1}(x) = 2x\, T_n(x) - T_{n-1}(x)$

whence $T_n(\cos\theta) = \cos(n\theta)$

Second Order Chebyshev Polynomials

$U_0(x) = 1$

$U_1(x) = 2x$

$U_2(x) = 4x^2 - 1$

$U_3(x) = 8x^3 - 4x$

$U_4(x) = 16x^4 - 12x^2 + 1$

$U_5(x) = 32x^5 - 32x^3 + 6x$

$U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1$

$U_7(x) = 128x^7 - 192x^5 + 80x^3 - 8x$

$U_8(x) = 256x^8 - 448x^6 + 240x^4 - 40x^2 + 1$

$U_0(x) = 1$

$U_1(x) = 2x$

$U_{n+1}(x) = 2x\, U_n(x) - U_{n-1}(x)$

whence $U_n(\cos\theta) = \sin(n+1)\theta \,/\, \sin\theta$

Spread Polynomials

$\sin^2(n\theta) = S_n(\sin^2\theta)$

$S_0(s) = 0$

$S_1(s) = s$

$S_2(s) = {}^{-}4s^2 + 4s$

$S_3(s) = 16s^3 - 24s^2 + 9s$

$S_4(s) = {}^{-}64s^4 + 128s^3 - 80s^2 + 16s$

$S_5(s) = 256s^5 - 640s^4 + 560s^3 - 200s^2 + 25s$

$S_6(s) = {}^{-}1024s^6 + 3072s^5 - 3456s^4 + 1792s^3 - 420s^2 + 36s$

$S_7(s) = 4096s^7 - 14336s^6 + 19712s^5 - 13440s^4 + 4704s^3 - 784s^2 + 49s$

$S_8(s) = {}^{-}16384s^8 + 65536s^7 - 106496s^6 + 90112s^5 - 42240s^4 + 10752s^3 - 1344s^2 + 64s$

TAUNTON SCHOOL

Counting

No.	***Greek***	***Latin***
1	*mono*	*uni*
2	*duo*	*bi*
3	*tri*	*tri*
4	*tetra*	*quad*
5	*penta*	*quin*
6	*hexa*	*sex*
7	*hepta*	*sept*
8	*octo*	*oct*
9	*nona*	*non*
10	*deca*	*dec*

These booklets are written and produced by Robert Goodhand

Although the formulae and expressions given have been individually derived and checked errors do creep in. The booklets are also continuously updated.

If you would like the latest issue, just email me at robert.goodhand@gmail.com