Mr. G's little booklet on

Trigonometrical (Hyperbolic) Functions

sinh , cosh , tanh , cosech , sech , coth
and inverses

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rg

Mr. G's Little Booklets are

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Trigonometric (Hyperbolic) Relationships I Key Identities

$$\cosh^2 x - \sinh^2 x = 1$$
$$\text{sech}^2 x = 1 - \tanh^2 x$$
$$\text{cosech}^2 x = \coth^2 x - 1$$

$$\tanh x = {}^{\sinh x}/_{\cosh x}$$
$$^{-}1 \leq \tanh x \leq {}^{+}1$$

Addition Formulae

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$
$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$ *note signs*

Double "angle" Formulae

$$\sinh 2x = 2 \sinh x \cosh x$$
$$\cosh 2x = 2 \sinh^2 x + 1$$
$$\cosh 2x = 2 \cosh^2 x - 1$$
$$\cosh 2x = {}^{1+\tanh^2 x}/_{1-\tanh^2 x}$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

Half "angle" Formulae

$$\sinh^2 x = \tfrac{1}{2}(\cosh 2x - 1)$$
$$\cosh^2 x = \tfrac{1}{2}(\cosh 2x + 1)$$
$$\tanh^2 x = {}^{\cosh 2x - 1}/_{\cosh 2x + 1}$$

note sign change from circular function

Product Formulae

remember $\sinh x = {}^{-}\sinh({}^{-}x)$

$$\sinh x \cosh y = \tfrac{1}{2}[\sinh(x+y) + \sinh(x-y)]$$
$$\cosh x \sinh y = \tfrac{1}{2}[\sinh(x+y) - \sinh(x-y)]^{\dagger}$$
$$\cosh x \cosh y = \tfrac{1}{2}[\cosh(x+y) + \cosh(x-y)]$$
$$\sinh x \sinh y = \tfrac{1}{2}[\cosh(x+y) - \cosh(x-y)]$$ *cf trig function*

Factor / Sum Formulae

set $x + y = a$ *set* $x - y = b$

then $x = \tfrac{1}{2}(a + b)$ *then* $y = \tfrac{1}{2}(a - b)$

$$\sinh a + \sinh b = 2 \sinh[\tfrac{1}{2}(a+b)] \cosh[\tfrac{1}{2}(a-b)]$$
$$\sinh a - \sinh b = 2 \cosh[\tfrac{1}{2}(a+b)] \sinh[\tfrac{1}{2}(a-b)]$$
$$\cosh a + \cosh b = 2 \cosh[\tfrac{1}{2}(a+b)] \cosh[\tfrac{1}{2}(a-b)]$$
$$\cosh a - \cosh b = 2 \sinh[\tfrac{1}{2}(a+b)] \sinh[\tfrac{1}{2}(a-b)]$$

Trigonometric (Hyperbolic) Relationships 2

define $\tanh x = \text{sinh x} / \text{cosh y}$

$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$ †

$\tanh(x-y) = \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$

$\tanh 2x = \frac{2\tanh x}{(1+\tanh^2 x)}$

$\sinh 2x = \frac{2\tanh x}{(1-\tanh^2 x)}$ *use these to solve*

$\cosh 2x = \frac{1+\tanh^2 x}{1-\tanh^2 x}$ sinh cosh *equations*

$\coth(x+y) = \frac{\coth y \coth x + 1}{\coth y + \coth x}$

$\coth(x-y) = \frac{\coth y \coth x - 1}{\coth y - \coth x}$

$\coth 2x = \frac{\coth^2 x + 1}{2\coth x}$

$\coth x = \text{cosech } 2x + \coth 2x$

$-\tanh x = \text{cosech } 2x - \coth 2x$

$\tanh x = \coth 2x - \text{cosech } 2x$

$\tanh x = 1 / (\text{cosech } 2x + \coth 2x)$ *cf* $\int \text{cosech } x \, dx$

$-\coth x = 1 / (\text{cosech } 2x - \coth 2x)$

$\coth x = 1 / (\coth 2x - \text{cosech } 2x)$

$\tanh x = \frac{\sinh 2x}{1 + \cosh 2x}$ $\quad \frac{(\cosh 2x - 1)}{\sinh 2x}$

$-\coth x = \frac{\sinh 2x}{1 - \cosh 2x}$ $\quad -\frac{(\cosh 2x + 1)}{\sinh 2x}$

$\coth x = \frac{\sinh 2x}{\cosh 2x - 1}$ $\quad \frac{(\cosh 2x + 1)}{\sinh 2x}$

$\sinh x \pm b \cosh x = R \sinh(x \pm c)$ $\quad a > b$

$\cosh x \pm b \sinh x = R \cosh(x \mp c)$

where $R \cosh c = a$ *and* $R \sinh c = b$

$R = \sqrt{(a^2 - b^2)}$ *note sign change*

Notes

† *Hence* $\tanh^{-1} x + \tanh^{-1} y = \tanh^{-1} \frac{x+y}{1+xy}$ *(nb the Lorentz transformation)*

so add [t] *relative velocities 0.9c + 0.8c is given by* $\tanh(\tanh^{-1}0.9 + \tanh^{-1}0.8) \Rightarrow 0.988c$

Trigonometric (Hyperbolic) Relationships 3

$$(1 + i\sinh x)/(1 - i\sinh x) = (\text{sech } x + i\tan x)^2 = \tanh^2(^-\tfrac{1}{2}x + \tfrac{1}{4}i\pi)$$

$$(1 - i\sinh x)/(1 + i\sinh x) = (\text{sech } x - i\tanh x)^2 = \coth^2(^-\tfrac{1}{2}x + \tfrac{1}{4}i\pi)$$

$$(\sinh x + 1)/(\sinh x - 1) = {}^-(\sec x + i\tan x)^2 = \coth^2(^-\tfrac{1}{2}x - \tfrac{1}{4}i\pi)$$

$$(\sinh x - 1)/(\sinh x + 1) = {}^-(\text{sech } x - i\tanh x)^2 = \tanh^2(^-\tfrac{1}{2}x - \tfrac{1}{4}i\pi)$$

$$(1 + \cosh x)/(1 - \cosh x) = {}^-(\text{cosech } x + \coth x)^2 = {}^-\coth^2(\tfrac{1}{2}x)$$

$$(1 - \cosh x)/(1 + \cosh x) = {}^-(\text{cosech } x - \coth x)^2 = {}^-\tanh^2(\tfrac{1}{2}x)$$

$$(\cosh x + 1)/(\cosh x - 1) = (\text{cosec } x + \cot x)^2 = \coth^2(\tfrac{1}{2}x)$$

$$(\cosh x - 1)/(\cosh x + 1) = (\text{cosec } x - \cot x)^2 = \tanh^2(\tfrac{1}{2}x)$$

$$(1 + \tanh x)/(1 - \tanh x) = \cosh 2x + \sinh 2x = e^{2x}$$

$$(1 - \tanh x)/(1 + \tanh x) = \cosh 2x - \sinh 2x = e^{-2x}$$

$$(\tanh x + 1)/(\tanh x - 1) = {}^-(\cosh 2x + \sinh 2x) = {}^-e^{2x}$$

$$(\tanh x - 1)/(\tanh x + 1) = {}^-(\cosh 2x - \sinh 2x) = {}^-e^{-2x}$$

$$\coth \tfrac{1}{2}x = \cosh x + \sinh x + 1 / \cosh x + \sinh x - 1 \quad \dagger$$

cf $\cot \tfrac{1}{2}x = i\,(\cos x + i\sin x + 1 / \cos x + i\sin x - 1)$

if $\sinh x = \tan y$

then $\cosh x = \sec y$

$\tanh x = \sin y$

Notes

† *this follows from the expression*

$$\tanh x = (e^{2x} - 1)/(e^{2x} + 1)$$

and let's take the opportunity to record

$$\cosh x = \tfrac{1}{2}(e^{x} + e^{-x})$$

$$\sinh x = \tfrac{1}{2}(e^{x} - e^{-x})$$

sech^2 *is a bell shaped distribution occurring in the natural world.*

Circular / Hyperbolic Relationships

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots \qquad x \in P$

let $e^x = \cosh x + \sinh x$

then $e^{-x} = \cosh x - \sinh x$ *because cosh is even and sinh is odd*

odd part $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} \ldots = {}^-i \sin ix$

even part $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} \ldots$ *a catenary* $= \cos ix$

hence $\tanh x = (e^x - e^{-x}) / (e^x + e^{-x}) = {}^-\tan ix$

and further $e^{ix} = \cos x + i \sin x$ *Euler's Theorem*

and $e^{-ix} = \cos x - i \sin x$

so $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix}) = \frac{1}{2i}(z - \frac{1}{z})$

$x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots$ *(an odd function)*

and $\cos x = \frac{1}{2}(e^{ix} + e^{-ix}) \qquad \frac{1}{2}(z + \frac{1}{z})$

$1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots$ *(an even function)*

also $i \tan x = (e^{2ix} - 1) / (e^{2ix} + 1)$

we define $e^x = \cosh x + \sinh x$

hence $e^{ix} = \cosh ix + \sinh ix$

compare this with $e^{ix} = \cos x + i \sin x$ *and substituting* x/i *for x we get*

$e^x = \cos ix - i\sin ix$

$\sinh ix$	$= i \sin x$	$\sin ix$	$= i \sinh x$
$\cosh ix$	$= \cos x$	$\cos ix$	$= \cosh x$
$\tanh ix$	$= i \tan x$	$\tan ix$	$= i \tanh x$
$\operatorname{cosech} ix$	$= {}^-i \operatorname{cosec} x$	$\operatorname{cosec} ix$	$= {}^-i \operatorname{cosech} x$
$\operatorname{sech} ix$	$= \sec x$	$\sec ix$	$= \operatorname{sech} x$
$\coth ix$	$= {}^-i \cot ix$	$\cot ix$	$= {}^-i \coth x$

nb $\sin^2 x = {}^-\sinh^2 ix$ *giving Osborne's Rule.*

Notes

Functions can be split into even and odd by $f(t) = \frac{1}{2}\{ f(t) + f({}^-t) \} + \frac{1}{2}\{ f(t) - f({}^-t) \}$

So $e^x = \frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x})$

Inverse Hyperbolic Functions 1

$\sinh^{-1}x$	=	$\ln\{x + \sqrt{(x^2 + 1)}\}$	$x \in$ P
$\text{cosech}^{-1}x$	=	$\ln\left(\frac{1}{x} + \sqrt{\left(\frac{1}{x^2} + 1\right)}\right)$	$x \neq 0$
or $\text{cosech}^{-1}x$	=	$\ln\left\{\frac{(1+\sqrt{(1 + x^2)})}{\lvert x \rvert}\right\}$	$x \neq 0$
$\cosh^{-1}x$	=	$\ln\{x \pm \sqrt{(x^2 - 1)}\}$	$x \geq 1$ *two valued*
$\text{sech}^{-1}x$	=	$\ln\left\{\frac{1}{x} + \sqrt{\left(\frac{1}{x} - 1\right)}\sqrt{\left(\frac{1}{x} + 1\right)}\right\}$	$0 < x \leq 1$
or $\text{sech}^{-1}x$	=	$\ln\left\{\frac{(1+\sqrt{(1 - x^2)})}{x}\right\}$	$0 < x \leq 1$
$\tanh^{-1}x$	=	$\frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right)$	$-1 < x < 1$ or $x^2 < 1$
$\coth^{-1}x$	=	$\frac{1}{2}\ln\left(\frac{x + 1}{x - 1}\right)$	$x \leq {}^-1$ *and* ${}^+1 \leq x$ or $x^2 \geq 1$
$\sinh^{-1}\frac{x}{a}$	=	$\ln\left(\frac{x + \sqrt{(x^2 + a^2)}}{a}\right)$	$x \in$ P
$\cosh^{-1}\frac{x}{a}$	=	$\ln\left(\frac{x + \sqrt{(x^2 - a^2)}}{a}\right)$	$x \geq 1$ *two valued*
$\tanh^{-1}\frac{x}{a}$	=	$\frac{1}{2}\ln\left(\frac{a + x}{a - x}\right)$	${}^-1 < x < {}^+1$ or $x^2 < 1$
$\coth^{-1}\frac{x}{a}$	=	$\frac{1}{2}\ln\left(\frac{x + a}{x - a}\right)$	$x \leq {}^-1$ *and* ${}^+1 \leq x$ or $x^2 \geq 1$

Relationship between Inv. Circular and Hyperbolic Functions

$\sin^{-1}ix$	=	$i\sinh^{-1}x$	$\sinh^{-1}ix$	=	$i\sin^{-1}x$
$\cos^{-1}ix$	=	$\pm i\cosh^{-1}x$	$\cosh^{-1}ix$	=	$\pm i\cos^{-1}x$
$\tan^{-1}ix$	=	$i\tanh^{-1}x$	$\tanh^{-1}ix$	=	$i\tan^{-1}x$
$\text{cosec}^{-1}ix$	=	${}^-i\,\text{cosech}^{-1}x$	$\text{cosech}^{-1}ix$	=	${}^-i\,\text{cosec}^{-1}x$
$\sec^{-1}ix$	=	$\pm i\,\text{sech}^{-1}x$	$\text{sech}^{-1}ix$	=	$\pm i\sec^{-1}x$
$\cot^{-1}ix$	=	${}^-i\coth^{-1}x$	$\coth^{-1}ix$	=	${}^-i\cot^{-1}x$

Notes

Inverse hyperbolic functions give the area of the sector of the unit hyperbola $x^2 - y^2 = 1$

The magnitudes are termed hyperbolic angles. May be written **ar**sinh *etc. (NOT* **arc**sinh*)*

$\cosh^{-1}x < \ln 2x < \sinh^{-1}x$ *. For x > 5 the difference between functions falls to 1%.*

$y = \cosh x$ *is known as the catenary in Statics.*

Inverse Hyperbolic Functions 2

if $\sinh b = {}^{1}/_{a}$ *then* $\operatorname{cosech} b = a$

hence $b = \sinh^{-1}({}^{1}/_{a})$ *and also* $b = \operatorname{cosech}^{-1} a$

hence $\operatorname{cosech}^{-1}(a) = \sinh^{-1}({}^{1}/_{a})$

$\sinh^{-1}x$ *is an odd function*

$$\sinh^{-1}(-x) = -\sinh^{-1}x \qquad x \in P$$

$$\operatorname{cosech}^{-1}(x) = \sinh^{-1}({}^{1}/_{x}) \qquad x \in P \text{ with discontinuity at } x = 0$$

$\cosh^{-1}x$ *is a two-valued function*

			$+1 \le x$	$+1 \le x$
$\cosh^{-1}(-x)$	=	$-\cosh^{-1}x + i\pi$	$x < -1$	$x < -1$
$\operatorname{sech}^{-1}(x)$	=	$\cosh^{-1}({}^{1}/_{x})$	$0 \le x \le +1$	$0 \le x \le +1$

$\tanh^{-1}x$ *is an odd function*

$$\tanh^{-1}(-x) = -\tanh^{-1}x \qquad -1 \le x \le +1$$

$$\coth^{-1}(x) = \tanh^{-1}({}^{1}/_{x}) \qquad x < -1 \text{ and } +1 < x$$

Inverse Hyperbolic Functions 3

$$\sinh^{-1}(-x) + \cosh^{-1}x = \tfrac{1}{2} i\pi \qquad x < 1$$

$$-\sinh^{-1}(-x) + \cosh^{-1}x = -\tfrac{1}{2} i\pi \qquad x > 1$$

$$\operatorname{cosec}^{-1}(-x) + \sec^{-1}x = -\tfrac{1}{2} i\pi \qquad 0 \le x \le +1$$

$$-\operatorname{cosec}^{-1}(-x) + \sec^{-1}x = \tfrac{1}{2} i\pi \qquad x < 0 \text{ and } 1 < x$$

$$\tanh^{-1}(-x) + \coth^{-1}x = \text{(doesn't exist in real domain)} \quad \dagger$$

$$\operatorname{cosech}^{-1}(-x) = \operatorname{cosech}^{-1}(x) \qquad x \in P \text{ with discontinuity at } x = 0$$

$$\operatorname{sech}^{-1}(-x) = i\pi - \operatorname{sech}^{-1}x \qquad \ddagger$$

$$\coth^{-1}(-x) = -\coth^{-1}(x) \qquad x < -1 \text{ and } +1 < x$$

Notes

All these relationships mirror their equivalent circular functions.

† *Algebraic manipulation might have misled you to conclude otherwise.*

‡ *A conjecture*

Complex Circular Functions

$$\sin(x \pm iy) = \sin x \cos iy \pm \cos x \sin iy$$
$$= \sin x \cosh y \pm i \cos x \sinh y$$
$$\cos(x \pm iy) = \cos x \cos iy \mp \sin x \sin iy$$
$$= \cos x \cosh y - i \sin x \sinh y$$
$$\tan(x \pm iy) = {}^{\tan x \pm \tan iy} / {}_{1 \mp \tan x \tan iy}$$
$$= {}^{\tan x \pm i \tanh y} / {}_{1 \mp i \tan x \tanh y}$$
$$= {}^{\tan x (1 \pm \tanh^2 y) \pm i \tanh y (1 + \tan^2 x)} / {}_{(1 + \tan^2 x \tanh^2 y)}$$

Complex Hyperbolic Functions †

$$\sinh(x \pm iy) = \sinh x \cosh iy \pm \cosh x \sinh iy$$
$$= \sinh x \cos y \pm i \cosh x \sin y$$
$$\cosh(x \pm iy) = \cosh x \cosh iy \pm \sinh x \sinh iy$$
$$= \cosh x \cos y \pm i \sinh x \sin y$$
$$\tanh(x \pm iy) = {}^{\tanh x \pm \tanh iy} / {}_{1 \pm \tanh x \tanh iy}$$
$$= {}^{\tanh x \pm i \tan y} / {}_{1 \pm i \tanh x \tan y}$$
$$= {}^{\tanh x (1 + \tan^2 y) \pm i \tan y (1 \pm \tanh^2 x)} / {}_{(1 + \tanh^2 x \tan^2 y)}$$

Periodicity of Hyperbolic Functions

$$e^{z + 2\pi i} = e^z \cdot e^{2\pi i}$$
$$= e^z(\cos 2\pi + i \sin 2\pi)$$
$$= e^z(1 + 0i)$$
$$= e^z$$

So as the functions cosh *and* sinh *are composed of the functions* e^z *and* e^{-z} *we can show*

$$\cosh(z + 2\pi i) = \cosh z$$
$$\sinh(z + 2\pi i) = \sinh z$$

and hence have periodicity $2\pi i$

Notes

† *need verifying*

Hyperbolic "Functions of Functions" Mapping

$\text{sech}(\text{sech}^{-1}x)$	=	x	$\cosh(\text{sech}^{-1}x)$	=	$^{1}/_{x}$
$\cosh(\cosh^{-1}x)$	=	x	$\text{sech}(\cosh^{-1})$	=	$^{1}/_{x}$
$\text{cosech}(\text{cosech}^{-1}x)$	=	x	$\sinh(\text{cosech}^{-1}x)$	=	$^{1}/_{x}$
$\sinh(\sinh^{-1}x)$	=	x	$\text{cosech}(\sinh^{-1}x)$	=	$^{1}/_{x}$
$\coth(\coth^{-1}x)$	=	x	$\tanh(\coth^{-1}x)$	=	$^{1}/_{x}$
$\tanh(\tanh^{-1}x)$	=	x	$\coth(\tanh^{-1}x)$	=	$^{1}/_{x}$
$\cosh(\sinh^{-1}x)$	=	$\sqrt{(x^2 + 1)}$ †	$\text{sech}(\sinh^{-1}x)$	=	$^{1}/_{\sqrt{(x^2+1)}}$
$\coth(\sinh^{-1}x)$	=	$^{\sqrt{(x^2+1)}}/_{x}$	$\tanh(\sinh^{-1}x)$	=	$^{x}/_{\sqrt{(x^2+1)}}$
$\sinh(\cosh^{-1}x)$	=	$\sqrt{(x^2 - 1)}$	$\text{cosech}(\cosh^{-1}x)$	=	$^{1}/_{\sqrt{(x^2-1)}}$
$\tanh(\cosh^{-1}x)$	=	$^{\sqrt{(x^2-1)}}/_{x}$	$\coth(\cosh^{-1}x)$	=	$^{x}/_{\sqrt{(x^2-1)}}$
$\tanh(\text{sech}^{-1}x)$	=	$\sqrt{(1 - x^2)}$	$\coth(\text{sech}^{-1}x)$	=	$^{1}/_{\sqrt{(1-x^2)}}$
$\text{sech}(\tanh^{-1}x)$	=	$\sqrt{(1 - x^2)}$	$\cosh(\tanh^{-1}x)$	=	$^{1}/_{\sqrt{(1-x^2)}}$
$\text{cosech}(\tanh^{-1}x)$	=	$^{\sqrt{(1-x^2)}}/_{x}$	$\sinh(\tanh^{-1}x)$	=	$^{x}/_{\sqrt{(1-x^2)}}$
$\sinh(\text{sech}^{-1}x)$	=	$^{\sqrt{(1-x^2)}}/_{x}$	$\text{cosech}(\text{sech}^{-1}x)$	=	$^{x}/_{\sqrt{(1-x^2)}}$
$\text{cosech}(\coth^{-1}x)$	=	$[(x-1)\sqrt{^{(x+1)}/_{(x-1)}}]$	$\sinh(\coth^{-1}x)$	=	*(inverse)*
$\text{sech}(\coth^{-1}x)$	=	$[(x-1)\sqrt{^{(x+1)}/_{(x-1)}}]/x$	$\cosh(\coth^{-1}x)$	=	*(inverse)*
$\coth(\text{cosech}^{-1}x)$	=	*messy*	$\tanh(\text{cosech}^{-1}x)$	=	$^{1}/_{\text{messy}}$
$\text{sech}(\text{cosech}^{-1}x)$	=	*messy*	$\cosh(\text{cosech}^{-1}x)$	=	$^{1}/_{\text{messy}}$

Notes

Taking the functions sinh cosh tanh and cosech sech coth we can perm into 36 relationships

If we arrange these by the result rather than the relationship the following pattern emerges

Personally I got a great deal of satisfaction in compiling this, which took some time!

† *by way of example* $\cosh(\sinh^{-1}x) = \sqrt{(x^2 + 1)} \Rightarrow \sinh^{-1}x = \cosh^{-1}\sqrt{(x^2 + 1)}$

Functions of Functions

$$\cosh(\sinh^{-1} x) = \sqrt{(x^2 + 1)} \qquad \cosh(\sinh^{-1} {}^{x}/_{a}) = {}^{1}/_{a}\sqrt{(x^2 + a^2)}$$

$$\sinh(\cosh^{-1} x) = \sqrt{(x^2 - 1)} \qquad \sinh(\cosh^{-1} {}^{x}/_{a}) = {}^{1}/_{a}\sqrt{(x^2 - a^2)}$$

$$\cosh(\ln x) = \tfrac{1}{2}(x + {}^{1}/_{x}) \qquad \cosh(a \ln x) = \tfrac{1}{2}(x^a + {}^{1}/_{x^a})$$

$$\sinh(\ln x) = \tfrac{1}{2}(x - {}^{1}/_{x}) \qquad \sinh(a \ln x) = \tfrac{1}{2}(x^a - {}^{1}/_{x^a})$$

$$\sin^{-1} \tanh x = \tan^{-1} \sinh x \qquad \sinh^{-1} \tan x = \tanh^{-1} \sin x$$

Periodicity of Circular and Hyperbolic Functions

The mapping between circular and hyperbolic functions is reflected in Osborne's rule.

Osborne's rule briefly is to replace $\sin^2$ *(or implied* $\sin^2$*) with* $-\sinh^2$

Strictly first to obtain the circular expansion in integral powers and then replace.

The hyperbolic functions are not periodic in the P *domain but period* $2\pi i$ *in* X *domain*

To verify hyperbolic equivalent of circular identity examine relationship in the complex plane

Further Identities

and finally two that you may not have seen before

$$\cosh^2 y + \sinh^2 x = \cosh(x + y)\cosh(x - y)$$

$$\cosh^2 y - \cosh^2 x = \sinh(x + y)\sinh(x - y)$$

Gudermannian Function

$$\operatorname{gd}(x) = {}_0\!\int^x \operatorname{sech} t \, dt$$

$$= 2\tan^{-1} \tanh \tfrac{1}{2}x \qquad = \sin^{-1} \tanh x$$

$$= 2\tan e^x - \tfrac{1}{2}\pi \qquad = \tan^{-1} \sinh x$$

Hence $\sin \operatorname{gd}(x) = \tanh x \qquad \operatorname{cosec} \operatorname{gd}(x) = \coth x$

$$\cos \operatorname{gd}(x) = \operatorname{sech} x \qquad \sec \operatorname{gd}(x) = \cosh x$$

$$\tan \operatorname{gd}(x) = \sinh x \qquad \cot \operatorname{gd}(x) = \operatorname{cosech} x$$

$$\operatorname{gd}^{-1}(\theta) = {}_0\!\int^\theta \sec t \, dt$$

$$= \ln|\sec\theta + \tan\theta| \qquad = \sinh^{-1} \tan\theta$$

$$= \tfrac{1}{2}\ln\left| {}^{1+\sin\theta}/_{1-\sin\theta} \right| \qquad = \tanh^{-1} \sin\theta$$

$$\ln(\tan(\tfrac{1}{2}\theta + \tfrac{1}{2}\pi))$$

Expansion cosh(nx) and sinh(nx) in power series.

These expansions follow the Chebyshev polynomials.

expansion cosh(nx) ~ n odd

Coefficients are Chebyshev Polynomials T(nx)

$$\cosh x = 1 \cosh x$$
$$\cosh 3x = 4 \cosh^3 x - 3 \cosh x$$
$$\cosh 5x = 16 \cosh^5 x - 20 \cosh^3 x + 5 \cosh x$$
$$\cosh 7x = 64 \cosh^7 x - 112 \cosh^5 x + 56 \cosh^3 x - 7 \cosh x$$

expansion cosh(nx) ~ n even

Coefficients are Chebyshev Polynomials T(nx)

$$\cosh 2x = 2 \cosh^2 x - 1$$
$$\cosh 4x = 8 \cosh^4 x - 8 \cosh^2 x + 1$$
$$\cosh 6x = 32 \cosh^6 x - 48 \cosh^4 x + 18 \cosh^2 x - 1$$
$$\cosh 8x = 128 \cosh^8 x - 256 \cosh^6 x + 160 \cosh^4 x - 32 \cosh^2 x + 1$$

expansion sinh(nx) ~ n odd

nb coefficients match cosh expansions except sign

$$\sinh x = 1 \sinh x$$
$$\sinh 3x = 4 \sinh^3 x + 3 \sinh x$$
$$\sinh 5x = 16 \sinh^5 x + 20 \sinh^3 x + 5 \sinh x$$
$$\sinh 7x = 64 \sinh^7 x + 112 \sinh^5 x + 56 \sinh^3 x + 7 \sinh x$$

expansion sinh(nx) ~ n even

$$\sinh 2x = 2 \sinh x \cosh x \,(1)$$
$$\sinh 4x = 2 \sinh x \cosh x \,(4 \sinh^2 x + 2)$$
$$\sinh 6x = 2 \sinh x \cosh x \,(16 \sinh^4 x + 16 \sinh^2 x + 3)$$
$$\sinh 8x = 2 \sinh x \cosh x \,(64 \sinh^6 x + 96 \sinh^4 x + 40 \sinh^2 x + 4)$$

Further Circular/Hyperbolic Function/Inverse Relationships

	Column A	Column B
	$\sinh x = -i \sin ix$	$\sin x = -i \sinh ix$
identical	$\sinh^{-1} x = -i \sin^{-1} ix$	$\sin^{-1} x = -i \sinh^{-1} ix$
	$\cosh x = \cos ix$	$\cos x = \cosh ix$
adds "± i"	$\pm i \cosh^{-1} x = \cos^{-1} x$??	$\pm i \cos^{-1} x = \cosh^{-1} x$??
	$\tanh x = -i \tan ix$	$\tan x = -i \tanh ix$
identical	$\tanh^{-1} x = -i \tan^{-1} ix$	$\tan^{-1} x = -i \tanh^{-1} ix$
	$\operatorname{cosech} x = i \operatorname{cosec} ix$	$\operatorname{cosec} x = i \operatorname{cosech} ix$
identical	$\operatorname{cosech}^{-1} x = i \operatorname{cosec}^{-1} ix$	$\operatorname{cosec}^{-1} x = i \operatorname{cosech}^{-1} ix$
	$\operatorname{sech} x = \sec ix$	$\sec x = \operatorname{sech} ix$
adds "± i"	$\pm i \operatorname{sech}^{-1} x = \sec^{-1} x$??	$\pm i \sec^{-1} x = \operatorname{sech}^{-1} x$??
	$\coth x = i \cot ix$	$\cot x = i \coth ix$
identical	$\coth^{-1} x = i \cot^{-1} ix$	$\cot^{-1} x = i \coth^{-1} ix$

Notes

To move from column A to column B just slide the "h" across

This is known as Goodhand's Rule (maybe).

Using the relationship for $\sin x$ *and* $\cos x$ *in terms of* $\sinh ix$ *and* $\cosh ix$ *we can thus derive all the equivalent hyperbolic relationships, by finally replacing* ix *by* x.

Osbourne's rule is immediately apparent.

When investigating $\int \sec x \, dx$ *, I determined*

$$\tan^{-1}(\sinh x) = \sin^{-1}(\tanh x)$$

I subsequently discovered this is the Gudermannian function $\operatorname{gd}(x)$ *, detailed on page 9.*

Functions of Inverse Functions

$\sinh(a \sinh^{-1} b)$

a b	1	2	3	4	5	7	10
1	1	2	3	4	5	7	10
2	√8	√80	√360	√1088	√2600	√9800	√40400
3	7	38	117	268	515	1393	4030
4	√288	√25920	√519840	√4739328	√27050400	√384199200	√6528801600
5	41	682	4443	17684	52525	275807	1620050

$\cosh(a \cosh^{-1} b)$ *compare with Chebyshev Polynomials* $T_a(b)$

a b	1	2	3	4	5	7	10
1	1	2	3	4	5	7	10
2	1	7	17	31	49	97	199
3	1	26	99	244	485	1351	3970
4	1	97	577	1921	4801	18817	79201
5	1	362	3363	15124	47525	262087	1580050

$\tanh(a \tanh^{-1} b)$

a b	0	0.2	0.3	0.4	0.5	0.7	0.9
1	0.10	0.200	0.3000	0.40000	0.500000	0.7000000	0.90000000
2	0.20	0.385	0.5505	0.68966	0.800000	0.9395973	0.99447514
3	0.29	0.543	0.7299	0.85405	0.928571	0.9890688	0.99970845
4	0.38	0.670	0.8449	0.93473	0.975610	0.9980622	0.99998465
5	0.46	0.767	0.9134	0.97150	0.991803	0.9996578	0.99999919

Notes

And this seems a good place to record the Bernoulli numbers.

$B_0 = 1$	$B_1 = {}^{-}½$	$B_2 = {}^{-1}/_{6}$	$B_3 = 0$	$B_4 = {}^{-1}/_{30}$	$B_5 = 0$
$B_6 = {}^{1}/_{42}$	$B_7 = 0$	$B_8 = {}^{-1}/_{30}$	$B_9 = 0$	$B_{10} = {}^{5}/_{66}$	$B_{11} = 0$
$B_{12} = {}^{-691}/_{2730}$	$B_{13} = 0$	$B_{14} = {}^{7}/_{6}$	$B_{15} = 0$	$B_{16} = {}^{-3617}/_{510}$	$B_{17} = 0$
$B_{18} = {}^{43867}/_{798?}$	$B_{19} = 0$	$B_{20} = {}^{?}/_{?}$	$B_{21} = 0$	$B_{21} = {}^{?}/_{?}$	$B_{23} = 0$

x	sinh	cosh	tanh	$sinh^{-1}$	$cosh^{-1}$	$tanh^{-1}$	ln (2x)
-5.0	-74.203	74.210	-1.000	-2.312	~	~	~
-4.5	-45.003	45.014	-1.000	-2.209	~	~	~
-4.0	-27.290	27.308	-0.999	-2.095	~	~	~
-3.5	-16.543	16.573	-0.998	-1.966	~	~	~
-3.0	-10.018	10.068	-0.995	-1.818	~	~	~
-2.5	-6.050	6.132	-0.987	-1.647	~	~	~
-2.0	-3.627	3.762	-0.964	-1.444	~	~	~
-1.5	-2.129	2.352	-0.905	-1.195	~	~	~
-1.0	-1.175	1.543	-0.762	-0.881	~	~	~
-0.9	-1.027	1.433	-0.716	-0.809	~	-1.472	~
-0.8	-0.888	1.337	-0.664	-0.733	~	-1.099	~
-0.7	-0.759	1.255	-0.604	-0.653	~	-0.867	~
-0.6	-0.637	1.185	-0.537	-0.569	~	-0.693	~
-0.5	-0.521	1.128	-0.462	-0.481	~	-0.549	~
-0.4	-0.411	1.081	-0.380	-0.390	~	-0.424	~
-0.3	-0.305	1.045	-0.291	-0.296	~	-0.310	~
-0.2	-0.201	1.020	-0.197	-0.199	~	-0.203	~
-0.1	-0.100	1.005	-0.100	-0.100	~	-0.100	~
0.0	0.000	1.000	0.000	0.000	~	0.000	~
0.1	0.100	1.005	0.100	0.100	~	0.100	-1.609
0.2	0.201	1.020	0.197	0.199	~	0.203	-0.916
0.3	0.305	1.045	0.291	0.296	~	0.310	-0.511
0.4	0.411	1.081	0.380	0.390	~	0.424	-0.223
0.5	0.521	1.128	0.462	0.481	~	0.549	0.000
0.6	0.637	1.185	0.537	0.569	~	0.693	0.182
0.7	0.759	1.255	0.604	0.653	~	0.867	0.336
0.8	0.888	1.337	0.664	0.733	~	1.099	0.470
0.9	1.027	1.433	0.716	0.809	~	1.472	0.588
1.0	1.175	1.543	0.762	0.881	0.000	~	0.693
1.5	2.129	2.352	0.905	1.195	0.962	~	1.099
2.0	3.627	3.762	0.964	1.444	1.317	~	1.386
2.5	6.050	6.132	0.987	1.647	1.567	~	1.609
3.0	10.018	10.068	0.995	1.818	1.763	~	1.792
3.5	16.543	16.573	0.998	1.966	1.925	~	1.946
4.0	27.290	27.308	0.999	2.095	2.063	~	2.079
4.5	45.003	45.014	1.000	2.209	2.185	~	2.197
5.0	74.203	74.210	1.000	2.312	2.292	~	2.303

Counting

No.	***Greek***	***Latin***
1	*mono*	*uni*
2	*duo*	*bi*
3	*tri*	*tri*
4	*tetra*	*quad*
5	*penta*	*quin*
6	*hexa*	*sex*
7	*hepta*	*sept*
8	*octo*	*oct*
9	*nona*	*non*
10	*deca*	*dec*

These booklets are written and produced by Robert Goodhand

Although the formulae and expressions given have been individually derived and checked errors do creep in. The booklets are also continuously updated.

If you would like the latest issue, just email me at robert.goodhand@gmail.com