Mr. G's little booklet on

Complex Numbers

Issue 5.0 4/16

Includes an introduction to hypercomplex numbers

rg

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Basic Relationships

$z = a + ib$ $\qquad z^* = a - ib$ *termed conjugate*

Continuous functions of x and y exist because each can be expressed individually viz.

$\mathcal{R}(z)$ *(ie a)* $= \frac{1}{2}(z + z^*)$ $\qquad z^* = z$ *iff* z *is real*

$\mathcal{I}(z)$ *(ie b)* $= \frac{1}{2i}(z - z^*)$ $\qquad z^* = {}^-z$ *iff* z *is imaginary*

$|z| = \sqrt{(a^2 + b^2)}$ *also termed* r *the modulus or absolute value*

and also $|z|^* = \sqrt{(a^2 + b^2)}$ $\qquad zz^* = a^2 + b^2$

$zz^* = |z|^2$ $\qquad$ *and* $^1/_z = {}^{z^*}/_{|z|^2}$

Definition of Operations with two complex numbers z and w

Add/Subt. $z \pm w = (a \pm c) + i(b \pm d)$

Mult.iplication $z \bullet w = (ac - bd) + i(bc + ad)$

Inversion $^1/_z = {}^a/_{a^2+b^2} - {}^{ib}/_{a^2+b^2} = {}^{(a - ib)}/_{a^2+b^2}$

Division $^z/_w = {}^{(ac + bd)}/_{(c^2 + d^2)} + i\,{}^{(bc - ad)}/_{(c^2 + d^2)}$

Square Root $\sqrt{z} = \pm\{\sqrt{[\frac{1}{2}(a + \sqrt{(a^2 + b^2)})]} + i\sqrt{[\frac{1}{2}({}^-a + \sqrt{(a^2 + b^2)})]}\}$

Commutativity of the Conjugate

$(z + w)^* = z^* + w^* = (a + c) - i(b + d)$

$(z \times w)^* = z^* \times w^* = (ac - bd) - i(bc + ad)$

$(^z/_w)^* = (^{z^*}/_{w^*}) = {}^{(ac+bd)}/_{(c^2+d^2)} - {}^{i(bc-ad)}/_{(c^2+d^2)}$

commutativity of conjugation holds for most functions eg $\sin z^* = (\sin z)^*$ *and* $(z^*)^n = (z^n)^*$

specifically $|z^*| = |z|^* = |z|$

Commutativity of the Modulus

Addition $|z + w| = |z| + |w|$ *if and only if* $z = kw$

Mult.iplication $|zw| = |z|\,|w| = \sqrt{(a^2c^2 + b^2d^2 + b^2c^2 + a^2d^2)}$

Division $|^z/_w| = {}^{|z|}/_{|w|}$ *algebraically this is a little miracle*

Square $|zw|^2 = |z|^2\,|w|^2 = (a^2c^2 + b^2d^2 + b^2c^2 + a^2d^2)$

Indices $|zw|^Y = |z|^Y\,|w|^Y$ *for* $Y \in X$

but $|z^Y w^Y| = |z|^Y\,|w|^Y$ *for* $Y \in P$ *because LHS must be real.*

Polar Coordinates $[r,\theta]$

$$r = \sqrt{(a^2 + b^2)}$$ *the modulus which we previously wrote as* $|z|$

$$\theta = \tan^{-1}({}^{b}/{}_{a})$$ *the amplitude, argument, or phase*

$$z = r(\cos\theta + i\sin\theta)$$

$$[r_1\theta_1][r_2\theta_2] = [r_1r_2(\theta_1+\theta_2)]$$

$$z_1z_2 = r_1r_2([\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$

$$z_1 / z_2 = r_1 / r_2([\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$$

$$[r,\theta]^{1/2} = [\sqrt{r},\tfrac{1}{2}\theta]$$

To obtain 2nd square root add 180° to θ.

$$[r,\theta]^{1/n} = [\sqrt[n]{r},{}^{\theta}/{}_{n}]$$

and the other roots are spaced symmetrically around the Argand diagram circle.

We can show $|z_1\,z_2| = |z_1|\,|z_2|$

$$|z_1 / z_2| = |z_1| / |z_2|$$

$$\arg(z_1\,z_2) = \arg z_1 + \arg z_2$$

$$\arg(z_1 / z_2) = \arg z_1 - \arg z_2$$

De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$$ *in cartesian coordinates*

$$[r,\theta]^n = [r^n,n\theta]$$ *in polar coordinates*

Proof of Euler's Identity

let $w = \cos\theta + i\sin\theta$ *nb I reserved* z *for* $r \neq 1$

let $dw = ({}^-\sin\theta + i\cos\theta)d\theta = i\,w\,d\theta$

$$\int {}^{dw}/{}_{w} = \int i\,d\theta$$

$$\ln w = i\,\theta + \text{constant}$$

at $\theta = 0$ $w = 1$ *so the constant of integration is zero and hence*

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Most famously $e^{i\pi} = {}^-1$

Multiplying a complex number by $e^{i\theta}$ *rotates it* θ *radians and it follows that* $e^{i2\pi} = 1$

Powers of i

i^1	i	i^{-1}	$-i$
i^2	-1	i^{-2}	-1
i^3	$-i$	i^{-3}	i
i^4	1	i^{-4}	1

$-i^1$	$-i$	$-i^{-1}$	i
$-i^2$	1	$-i^{-2}$	1
$-i^3$	i	$-i^{-3}$	$-i$
$-i^4$	-1	$-i^{-4}$	-1

$(-i)^1$	$-i$	$(-i)^{-1}$	i
$(-i)^2$	-1	$(-i)^{-2}$	-1
$(-i)^3$	i	$(-i)^{-3}$	$-i$
$(-i)^4$	1	$(-i)^{-4}$	1

de Moivre's Theorem Extended

$$\{r(\cos\theta + i\sin\theta)\}^n = r^n(\cos n\theta + i\sin n\theta)$$

$$\cos\theta + i\sin\theta = \text{cis}\,\theta$$

$$\cos\theta - i\sin\theta \qquad {}^1/_{\text{cis}\,\theta}$$

$$\text{cis}^n\theta = \text{cis}\,n\theta \qquad n \in P$$

Euler's Theorem Extended

$$r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$r(\cos\theta - i\sin\theta) = re^{-i\theta}$$

$$z_1\ z_2 = r_1\ r_2 e^{i(\theta 1 + \theta 2)}$$

$$z_1 / z_2 = r_1 / r_2 e^{i(\theta 1 - \theta 2)}$$

$$z^n = \cos n\theta + i\sin n\theta$$

and $z^{-n} = \cos n\theta - i\sin n\theta$ *follows immediately*

Hence we find

$$z^n + z^{-n} = 2\cos n\theta$$

$$z^n - z^{-n} = 2i\sin n\theta$$

n^{th} Roots of Unity - Method A

if $w^n = 1$ *then* $w = 1^{1/n}$

so $w = [1, 2\pi k]^{1/n}$

$w = [1, {}^{2\pi k}/_n]$

$w = e^{2\pi ki/n}$ $n \in '$

n^{th} Roots of Unity - Method B

if $w = e^{i\theta}$

then $w = (\cos\theta + i\sin\theta)$

$w = \text{cis}\,\theta$

so the n^{th} *roots of* $w = e^{2\pi ki/n}$ $n \in '$

3^{rd} Roots of Unity forming a multiplicative group

$e^{2\pi ki/3} = \cos({}^{2\pi k}/_3) + i\sin({}^{2\pi k}/_3)$

×	1	ω	ω²
1	1	ω	ω²
ω	ω	ω²	1
ω²	ω²	1	ω

$k = 1$	ω	=	$^{-}0.5 + 0.8660\,i$
$k = 2$	ω^2	=	$^{-}0.5 - 0.8660\,i$
$k = 3$	ω^3	=	1

Multiplying by ω *rotates the triangle representing the 3 roots by 120° anticlockwise.*

5^{th} Roots of Unity forming a multiplicative group

$e^{2\pi ki/5} = \cos({}^{2\pi k}/_5) + i\sin({}^{2\pi k}/_5)$

×	1	ε	ε²	ε³	ε⁴
1	1	ε	ε²	ε³	ε⁴
ε	ε	ε²	ε³	ε⁴	1
ε²	ε²	ε³	ε⁴	1	ε
ε³	ε³	ε⁴	1	ε	ε²
ε⁴	ε⁴	1	ε	ε²	ε³

$k = 1$	ε	=	$0.3090 + 0.9511\,i$
$k = 2$	ε^2	=	$^{-}0.8090 + 0.5878\,i$
$k = 3$	ε^3	=	$^{-}0.8090 - 0.5878\,i$
$k = 4$	ε^4	=	$0.3090 - 0.9511\,i$
$k = 5$	ε^5	=	1

Multiplying by ε *rotates the pentagon representing the 5 roots by 72° anticlockwise.*

Notes

Just remember $e^{2\pi in} = 1$ for $n \in Z$ *For* $w^n = 1$ *then* $1 + w + w^2 + w^3 \ldots w^{n-1} = 0$

Complex Logs Method A

$$z = re^{i\theta}$$

$$z = e^{\ln r}e^{i\theta}$$

Hence $\text{Ln } z = \ln r + i\theta$ *where* $r = |z|$

But remembering we can add any integer multiple of $2\pi i$ *to* $\text{Ln } z$ *so the general solution is*

$$\text{Ln } z = \ln|z| + i(\theta + 2k\pi) \qquad \theta = \arg z$$

$$\text{Ln}(^{-}z) = \ln|z| + i\{\theta + (2k+1)\pi\}$$

Complex Logs Method B

These relationships for complex logs can also be deduced directly using Euler's Theorem

let $a + ib = e^{\alpha + i\beta}$

$$= e^{\alpha}e^{i\beta}$$

$$= e^{\alpha}(\cos\beta + i\sin\beta)$$

Hence $a = e^{\alpha}\cos\beta$

and $b = e^{\alpha}\sin\beta$

squaring a and b and rearranging we again deduce $\alpha = \frac{1}{2}\ln(a^2 + b^2)$ *and* $\beta = \tan^{-1}{}^{b}/_{a}$

Some Principle Values

$\text{Ln}(^{-}a) = \ln a + i\pi$ $\qquad a \in P^{+}$

$\text{Ln}(^{-}1) = i\pi$ *which immediately gives* $e^{i\pi} = {}^{-}1$

$\text{Ln}(i) = \frac{1}{2}i\pi$ $\qquad \text{Ln}(^{-}i) = {}^{-}\frac{1}{2}i\pi$

$i^{i} = e^{i\ln i} = e^{-\frac{1}{2}\pi}$

Summary

We now have a procedure to determine logs of complex numbers such that

$\text{Ln}(wz) = \text{Ln } w + \text{Ln } z$ *where* w *and* $z \in X$

and $b^{w+z} = b^{w}b^{z}$

We have already established $a^{x} = e^{x\ln a}$ *so now we can write* $w^{z} = e^{z\,\text{Ln } w}$

But we can add any integer multiple of $2\pi i$ *to* $\text{Ln } w$ *so we can* $\times\div$ w^{z} *by* $e^{z\,2\pi i}$

Complex Powers - Method A *(sans Euler except for the last bit!)*

$$a + ib = r[\cos\theta + i\sin\theta]$$

$$(a + ib)^c = r^c[\cos\theta + i\sin\theta]^c$$

$$= r^c[\cos(c\theta) + i\sin(c\theta)] \quad \textit{by De Moivre}$$

next $(a + ib)^{id} = r^{id}(\cos\theta + i\sin\theta)^{id}$

$$e^{id\ \ln r}[\cos(id\theta) + i\sin(id\theta)]$$

$$= [e^{i\ id\theta}][e^{id\ \ln r}] \quad \textit{swapped around}$$

$$= [e^{-d\theta}][\cos(d\ \ln r) + i\sin(d\ \ln r)] \quad \textit{nb } d\theta \textit{ is not calculus!}$$

nb the "r" term becomes Euler's equation while the original Euler equ. becomes the "r" term.

so consider $(a + ib)^c = A + iB$ *and* $(a + ib)^{id} = C + iD$

hence $(a + ib)^{c + id} = AC - BD + i(BC + AD)$

$$\mathcal{R}[(a + ib)^{c + id}] = r^c e^{-d\theta}[\cos(c\theta)\cos(d\ \ln r) - \sin(c\theta)\sin(d\ \ln r)]$$

$$= r^c e^{-d\theta}[\cos(c\theta + d\ \ln r)]$$

$$\mathcal{I}[(a + ib)^{c + id}] = r^c e^{-d\theta}[\sin(c\theta)\cos(d\ \ln r) + \cos(c\theta)\sin(d\ \ln r)]$$

$$= r^c e^{-d\theta}[\sin(c\theta + d\ \ln r)]$$

hence $(a + ib)^{c + id} = r^c e^{-d\theta}\{[\cos(c\theta + d\ \ln r)] + i[\sin(c\theta + d\ \ln r)]\}$

$$= r^c e^{-d\theta} e^{i(c\theta + d\ \ln r)} \quad \textit{neat!}$$

Complex Powers - Method B

$$a + ib = re^{i\theta}$$

so consider $(a + ib)^c = r^c e^{ic\theta}$ *and* $(a + ib)^{id} = r^{id}e^{-d\theta}$

switching around $= e^{-d\theta}e^{id\ \ln r}$

hence $(a + ib)^{c + id} = r^c e^{-d\theta} e^{i(c\theta + d\ \ln r)}$ *as above*

and written out in full

$$\{[\sqrt{(a^2+b^2)}]^c \bullet e^{-d \arctan b/a}\}\{\cos[c \tan^{-1} {}^b/_a + d \ln\sqrt{(a^2+b^2)}] + i\sin[c \tan^{-1} {}^b/_a + d \ln\sqrt{(a^2+b^2)}]\}$$

Complex Powers - Method C - Using Complex Logs

$(a + ib)^{c + id} = e^{(c + id)(\ln r + i\theta)}$ *from the work done on complex logs*

$$= e^{c \ln r + ic\theta + id \ln r - d\theta}$$

$$= r^c e^{-d\theta} e^{i(c\theta + d\ \ln r)} \quad \textit{as above}$$

The Complex Exponential Series

Let $\exp(z) = 1 + z + {}^{z^2}/_{2!} + {}^{z^3}/_{3!} \ldots$

where $z = r(\cos\theta + i\sin\theta)$

now $\exp(r) = 1 + r + {}^{r^2}/_{2!} + {}^{r^3}/_{3!} \ldots$ *converges for* $r \in P$

Hence the series for z *converges absolutely for all values of* z

Now to determine if the sum is of the form e^z *we need to investigate* $\exp(w) \times \exp(z)$

$$\exp(w) \times \exp(z) = (1 + z + {}^{z^2}/_{2!} + {}^{z^3}/_{3!} \ldots)(1 + z + {}^{z^2}/_{2!} + {}^{z^3}/_{3!} \ldots)$$

and multiplying out directly and ignoring for now terms higher than the third degree we get

$$\exp(w) \times \exp(z) = 1 + w + z + {}^{w^2}/_{2!} + wz + {}^{z^2}/_{2!} + {}^{w^3}/_{3!} + {}^{w^2z}/_{2!} + {}^{wz^2}/_{2!} + {}^{z^3}/_{3!}$$

$$= 1 + (w + z) + {}^{(w+z)^2}/_{2!} + {}^{(w+z)^3}/_{3!}$$

$$\exp(w) \times \exp(z) = \exp(w+z)$$

When $z = 0$ we have $\exp(0) = 1$ and when $z = 1$ *we have* $\exp(1) = e$

Further $\exp(z) \times \exp(z) = \exp(2z)$ *and from above we determine* $\exp(2) = e^2$.

This argument can be extended to m terms and we may safely assume $\exp(z) = e^z$

when $z = i\theta$

$$e^{i\theta} = 1 + i\theta + {}^{i^2\theta^2}/_{2!} + {}^{i^3\theta^3}/_{3!} \ldots$$

$$= 1 + i\theta - {}^{\theta^2}/_{2!} - {}^{i\theta^3}/_{3!} + {}^{\theta^4}/_{4!}$$

$$= 1 - {}^{\theta^2}/_{2!} + {}^{\theta^4}/_{4!} \ldots) + i(q - {}^{\theta^3}/_{3!} + {}^{\theta^5}/_{5!} \ldots)$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

Hence $\cos\theta = \tfrac{1}{2}(e^{i\theta} + e^{-i\theta})$

and $\sin\theta = \tfrac{1}{2i}(e^{i\theta} - e^{-i\theta})$

further $e^{i\theta} = \cosh i\theta + \sinh i\theta$ *by definition*

so $\cos\theta = \cosh i\theta$

and $i\sin\theta = \sinh i\theta$

Notes

The above is not a formal proof $\exp(z) = e^z$ *but a strong indicator.*

Convergence of Complex Series

For the complex power series $a_0 + a_1 z + a_2 z^2 + a_3 z^3 \ldots$

there exists in the complex plain a circle of convergence centre $z = 0$

$$^{1}/_{1-z^2} = 1 + z^2 + z^4 + z^6 + z^8 \ldots \qquad \text{Has "poles" at } \pm 1$$

$$^{1}/_{1+z^2} = 1 - z^2 + z^4 - z^6 + z^8 \ldots \qquad \text{Has "poles" at } \pm 1$$

If z lies inside the circle the series converges and outside diverges.

The Mandelbrot set is the "circle" of convergence for c in the relationship $z \rightarrow z^2 + c$

Mapping

$z \rightarrow z + w$ *is a translation in the complex plane sending* 0 *to* w.

$z \rightarrow zw$ *is a rotation and expansion (or contraction) sending* 1 *to* w.

The special case $z \rightarrow iz$ *is a rotation 90° anticlockwise.*

Representation Complex Numbers by Matrices

let $z = a + ib$ *be represented by* $a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ †

Hence $z = a + ib$

$= \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

So $z^* = a - ib$

$= \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

$= \begin{bmatrix} a & -b \\ b & a \end{bmatrix}^T$

so the conjugate is represented by the transpose matrix.

Further $|z|^2 = a^2 + b^2 = \begin{vmatrix} a & -b \\ b & a \end{vmatrix}$

so the modulus ² is represented by the determinant.

Finally we calculate $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}^{-1} = {}^{1}/_{a^2+b^2} \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

$= {}^{a - ib}/_{a^2+b^2} = z^{-1}$ *as expected*

Notes

† *These are the matrices* **I** *and* $\mathbf{I}_{rot-90}$

Algebras - Number Systems

		Commutative $ab = ba$	*Associative* $a(bc) = (ab)c$	*Division Algebra* $\lvert xy\rvert = \lvert x\rvert\,\lvert y\rvert$	*Alternate* $a(ab) = (aa)b$	*Power Associative* x^n *defined*
Real	$\mathbb{P}$	Yes	Yes	Yes	Yes	Yes
Complex	$\mathbb{X}$	Yes	Yes	Yes	Yes	Yes
Tessarines	$\mathbb{T}$	Yes	Yes	Yes	Yes	Yes
Quarternions	$\mathbb{H}$	No	Yes	Yes	Yes	Yes
Octonions	$\mathbb{O}$	No	No	Yes	Yes	Yes
Sedenions	Σ	No	No	No †	No	Yes

Except tessarines, all these Algebras have an identity element and a multiplicative inverse.
Sedenions are not a division algebra because of elements which when multplied together = 0.
$\mathbb{T}$ *is termed a bi-complex alegebra and* $\mathbb{H}\ \mathbb{O}\ \Sigma$ *(with* $\mathbb{T}$*) are termed hypercomplex algebras.*

for $\mathbb{X}$ $z = x + iy$ $\qquad i = \sqrt{-1} \qquad x \in \mathbb{P}$

for $\mathbb{T}$ $t = w + xi + yj + zk = (w + xi) + (y + zi)$ since $ij = k$

so if $t = p + q$ then $t \;\alpha \begin{pmatrix} p & q \\ q & p \end{pmatrix}$

$ik = i(ij) = (ii)j = -j$

commutative $\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

inverse *because the modulus is zero the inverse does not exist*

for $\mathbb{H}$ $z = v + iw + jx + ky$

$i^2 = j^2 = k^2 = ijk = -1$

conjugate $q^* = -\tfrac{1}{2}(q + iqi + jqj + kqk)$

norm $\lvert q\rvert = \sqrt{(qq^*)} = \sqrt{(v^2 + w^2 + x^2 + y^2)}$

$\lvert pq\rvert = \lvert p\rvert\,\lvert q\rvert$ $\qquad$ *hence a division algebra*

non-comm. $ij = k$ *but* $ji = -k$

inverse $q^{-1} = q^*/\lvert q\rvert^2$

for $\mathbb{O}$ $x = x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5 + x_6e_6 + x_7e_7$

conjugate $x^* = x_0e_0 - x_1e_1 - x_2e_2 - x_3e_3 - x_4e_4 - x_5e_5 - x_6e_6 - x_7e_7$

$(xy)^* = y^* x^*$ $\qquad$ *note change of order*

norm $\lvert x\rvert = \sqrt{(x^*x)}$

$\lvert x\rvert^2 = x^*x = x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$

non-comm. $e_ie_j = -e_je_i$

non-associative $(e_ie_j)e_k) = -e_i(e_je_k)$

Tessarines

×	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	k	1	i
k	k	$-j$	i	-1

Octonions

×	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

There are 480 possible octonion tables

Quarternions

×	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

Row left factor column right factor

×	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6	e_9	$-e_8$	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$	e_{10}	e_{11}	$-e_8$	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$	e_{11}	$-e_{10}$	e_9	$-e_8$	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3	e_{12}	e_{13}	e_{14}	e_{15}	$-e_8$	$-e_9$	$-e_{10}$	$-e_{11}$
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	$-e_8$	e_{11}	$-e_{10}$
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	$-e_8$	e_9
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	$-e_8$
e_8	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_{12}$	$-e_{13}$	$-e_{14}$	$-e_{15}$	-1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_9	e_9	e_8	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$	$-e_1$	-1	$-e_3$	e_2	$-e_5$	e_4	e_7	$-e_6$
e_{10}	e_{10}	e_{11}	e_8	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}	$-e_2$	e_3	-1	$-e_1$	$-e_6$	$-e_7$	e_4	e_5
e_{11}	e_{11}	$-e_{10}$	e_9	e_8	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}	$-e_3$	$-e_2$	e_1	-1	$-e_7$	e_6	$-e_5$	e_4
e_{12}	e_{12}	e_{13}	e_{14}	e_{15}	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_4$	e_5	e_6	e_7	-1	$-e_1$	$-e_2$	$-e_3$
e_{13}	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	e_8	e_{11}	$-e_{10}$	$-e_5$	$-e_4$	e_7	$-e_6$	e_1	-1	e_3	$-e_2$
e_{14}	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	e_8	e_9	$-e_6$	$-e_7$	$-e_4$	e_5	e_2	$-e_3$	-1	e_1
e_{15}	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	e_8	$-e_7$	e_6	$-e_5$	$-e_4$	e_3	e_2	$-e_1$	-1

Greek Alphabet				Principle/Simplest Use	English	Type
alpha	Α	*not used*	α	*first root of quadratic*	a	**a**
beta	Β	*Beta function*	β	*second root of quadratic*	b	**b**
gamma	Γ	*Gamma function*	γ	*Euler's constant*	g	**g**
delta	Δ	*Difference operator*	δ	*small increment*	d	**d**
epsilon	Ε	*not used*	ε	*error*	short e	**e**
zeta	Ζ	*not used*	ζ	*Riemann zeta function*	z	**z**
eta	Η	*not used*	η	*efficiency*	long e	**h**
theta	Θ	*asymp. tight bound*	θ	*angle*	th	**q**
iota	Ι	*not used*	ι	*imaginary unit*	i	**i**
kappa	Κ	*not used*	κ	*curvature*	k	**k**
lambda	Λ	*diag. matrix eigen-values*	λ	*failure rate*	l	**l**
mu	Μ	*not used*	μ	*population mean*	m	**m**
nu	Ν	*not used*	ν	*poisson ratio*	n	**n**
xi	Ξ	*grand canonical ensemble*	ξ	*damping coefficient*	x	**x**
omicron	Ο	*limiting behaviour function*	ο	*generally not used*	short o	**o**
pi	Π	*Product operator*	π	*ratio* $^c/_d$ *circle*	p	**p**
rho	Ρ	*not used*	ρ	*correlation coefficient*	r	**r**
sigma	Σ	*summation*	σ	*standard deviation*	s	**s**
tau	Τ	*not used*	τ	*mean lifetime*	t	**t**
upsilon	Υ	*Bessel function*	υ	*generally not used*	u	**u**
phi	Φ	*cumulative function*	ϕ	*golden ratio*	ph	**f**
phi (alt.)	ϑ	*not used*	φ	***normal function*** ***scalar potential***	ph	**j**
chi	Χ	*probability function*	χ^2	*chi-squared prob.function*	ch	**c**
psi	Ψ	*not used*	ψ	*wave function*	ps	**y**
omega	Ω	*mathematical constant*	ω	*angular frequency*	long o	**w**
stigma	ς					**v**
pomega			ϖ	*angular velocity*		**v**

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Orders of Magnitude

septillionth	yocto-	y	10^{-24}	septillion	yotta-	Y	10^{24}
sextillionth	zepto-	z	10^{-21}	sextillion	zetta-	Z	10^{21}
quintillionth	atto-	a	10^{-18}	quintillion	exa-	E	10^{18}
quadrillionth	femto-	f	10^{-15}	quadrillion	peta-	P	10^{15}
trillionth	pico-	p	10^{-12}	trillion	tera-	T	10^{12}
billionth	nano-	n	10^{-9}	billion	giga-	G	10^{9}
millionth	micro-	μ	10^{-6}	million	mega-	M	10^{6}
thousandth	milli-	m	10^{-3}	thousand	kilo-	k	10^{3}
hundredth	centi-	c	10^{-2}	hundred	hecto-	h	10^{2}
tenth	deci-	d	10^{-1}	ten	deca-	da	10^{1}
one	-	-	10^{0}	one	-	-	10^{0}

Mathematical Constants - 30 decimals (last place not rounded)

pi	π	=	*3.14159 26535 89793 23846 26433 83279...*
exponential	e	=	*2.71828 18284 59045 23536 02874 71352...*
Pythagoras's	$\sqrt{2}$	=	*1.41421 35623 73095 04880 16887 24209...*
	$\sqrt{3}$	=	*1.73205 08075 68877 29352 74463 41505...*
	$\log 2$	=	*0.69314 71805 59945 30941 72321 21458...*
golden ratio	ϕ	=	*1.61803 39887 49894 84820 45868 34365...*
Euler-Mascheroni	γ	=	*0.57721 56649 01532 86060 65120 90082...*
Feigenbaum's	δ	=	*4.66920 16091 02990 67185 32038 20466...*
	$\xi(2)$	=	*1.64493 40668 48226 43647 24151 66646...*
Apery's	$\xi(3)$	=	*1.20205 69031 59594 28539 97381 61511...*
	$\xi(4)$	=	*1.08232 32337 11138 19151 60036 96541...*
Euler's	$\xi(5)$	=	*1.03692 77551 43369 92633 13654 86457...*
	$\xi(6)$	=	*1.01734 30619 84449 13971 45179 29790...*
	e^{π}	=	*23.14069 26327 79269 00572 90863 67948...*

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Prime Numbers (in columns of 25)

2	101	233	383	547	701	877	1049	1229	1429	1597	1783
3	103	239	389	557	709	881	1051	1231	1433	1601	1787
5	107	241	397	563	719	883	1061	1237	1439	1607	1789
7	109	251	401	569	727	887	1063	1249	1447	1609	1801
11	113	257	409	571	733	907	1069	1259	1451	1613	1811
13	127	263	419	577	739	911	1087	1277	1453	1619	1823
17	131	269	421	587	743	919	1091	1279	1459	1621	1831
19	137	271	431	593	751	929	1093	1283	1471	1627	1847
23	139	277	433	599	757	937	1097	1289	1481	1637	1861
29	149	281	439	601	761	941	1103	1291	1483	1657	1867
31	151	283	443	607	769	947	1109	1297	1487	1663	1871
37	157	293	449	613	773	953	1117	1301	1489	1667	1873
41	163	307	457	617	787	967	1123	1303	1493	1669	1877
43	167	311	461	619	797	971	1129	1307	1499	1693	1879
47	173	313	463	631	809	977	1151	1319	1511	1697	1889
53	179	317	467	641	811	983	1153	1321	1523	1699	1901
59	181	331	479	643	821	991	1163	1327	1531	1709	1907
61	191	337	487	647	823	991	1171	1361	1543	1721	1913
67	193	347	491	653	827	1009	1181	1367	1549	1723	1831
71	197	349	499	659	829	1013	1187	1373	1553	1733	1933
73	199	353	503	661	839	1019	1193	1381	1559	1741	1949
79	211	359	509	673	853	1021	1201	1399	1567	1747	1951
83	223	367	521	677	857	1031	1213	1409	1571	1753	1973
89	227	373	523	683	859	1033	1217	1423	1579	1759	1979
97	229	379	541	691	863	1039	1223	1427	1583	1777	1999

Notes

Prime Number Theorem states that the number of primes up to n, $\pi_n \sim {}^{n}/_{\ln(n)}$

Alternatively the nth *prime number* $p_n \sim n \ln(n)$. *So* $p_{300} \sim 300 \ln 300 = 1711$ *(cf* 1999*)*

If $\mathrm{li} = \int {}^{dt}/_{\ln t}$ *then* $\mathrm{Li}(x) = {}_2\int^x {}^{dt}/_{\ln t} = \mathrm{li}(x) - \mathrm{li}(2)$ *is a better approximation to* $\pi(x)$

Goodhand's conjecture states the percent proportion of primes approximately equals the percent that $^{n}/_{\ln(n)}$ *underestimates p(n). Hence* $\pi(n)$ *better* $\approx \frac{1}{2}(1-\sqrt{\{1 - {}^{4}/_{\ln(n)}\}}$

TAUNTON SCHOOL

Counting

No.	*Greek*	*Latin*
1	*mono*	*uni*
2	*duo*	*bi*
3	*tri*	*tri*
4	*tetra*	*quad*
5	*penta*	*quin*
6	*hexa*	*sex*
7	*hepta*	*sept*
8	*octo*	*oct*
9	*nona*	*non*
10	*deca*	*dec*

These booklets are written and produced by Robert Goodhand

Although the formulae and expressions given have been individually derived and checked errors do creep in. The booklets are also continuously updated.

If you would like the latest issue, just email me at robert.goodhand@gmail.com

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