*Mr. G's little booklet on*

# Differentiation and Integration Standard Forms

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## Mr. G's Little Booklets are

1 **Symbols and Definitions**

2 **Circular Functions**

3 **Hyperbolic Functions**

4 **Complex Numbers**

5 **Calculus**

6 **Series**

7 **Venn Diagrams**

8 **Logic and Propositional Calculus**

9 **Vectors and Matrices**

10 **Probability**

11 **Laplace and Fourier Transforms**

12 **Miscellaneous Aspects of Mathematics**

13 **Statistical Tables**

14 **Trigonometric and Logarithmic Tables**

15 **Investigations - General**

16 **Investigations - Number**

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## Definition of Differentiation

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} \quad \dagger$$

*and it follows that* $\frac{dy}{dx} = \frac{1}{dx/dy}$

## Chain Rule *(function of a function $f(x)$ and $g(x)$ )*

$$\frac{d}{dx}[f\ g(x)] = f'[g(x)] \cdot g'(x)$$

*or more simply* $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

## Product Rule *(Leibnitz Rule)*

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

## Generalised Power Rule *(where u and v are functions of x)*

$$\frac{d}{dx}(u^v) \quad vu^{(v-1)} \cdot \frac{du}{dx} + u^v \cdot \ln u \cdot \frac{dv}{dx}$$

*To derive take logs of both sides and then differentiate using the Product Rule*

## Power Rule *(where $u = x$ and $v = n$ )*

$$\frac{d}{dx}(x^n) = n\,x^{n-1}$$

*The power rule can be derived directly from logarithmic differentiation* $\frac{d}{dx} \ln x^n$

## Quotient Rule *(derived from the Product Rule)*

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \left[v\frac{du}{dx} - u\frac{dv}{dx}\right] / v^2$$

## Reciprocal Rule *(case of Quotient Rule $u = 1$)*

$$\frac{d}{dx}\left(\frac{1}{v}\right) = \frac{-dv/dx}{v^2}$$

## Logarithmic Rule *(case of Chain rule where $u = \ln x$)*

$$\frac{d}{dx}[\ln f(x)] = \frac{f'(x)}{f(x)}$$

## Stationary Points

*if* $\frac{dy}{dx} = 0$ *and* $\frac{d^2y}{dx^2} > 0$ *then we have a minimum point*

*if* $\frac{dy}{dx} = 0$ *and* $\frac{d^2y}{dx^2} < 0$ *then we have a maximum point*

*if* $\frac{dy}{dx} = 0$ *and* $\frac{d^2y}{dx^2} = 0$ *then the situation is indeterminate*

## Notes

† *Non standard analysis of hyperreals legitimises the infinitesimal* dx *without using limits.*



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## Integration - Primary Methods

| | | | |
|---|---|---|---|
| $\int (u + v + w)\,dx$ | = | $\int u\,dx + \int v\,dx + \int w\,dx$ | *terms can be integrated separately* |
| $\int a\, f(x)\,dx$ | = | $a \int f(x)\,dx$ | *constants unaffected by integration* |
| $\int f(x)\,dx$ | = | $\int f(u)\, {}^{dx}/_{du}\, du$ | *Integration by substitution* |
| $\int u\,dv$ | = | $uv - \int v\,du$ | *Integration by parts* † |

*Let* u *be the easy differential and* v *be the easy integral.*

*If there is just an unintegrable function* u *then multiply by 1 (strictly you set* $dv = 1 \cdot dx$ *).*

| | | | |
|---|---|---|---|
| $\int f'(x)[f(x)]^n dx$ | = | $({}^{1}/_{n+1})\,[f(x)]^{n+1}$ | *when integral is of this pattern* |

## Trigonometric Identities

*To integrate* $\sin^2 x\, dx$ *or* $\cos^2 x\, dx$ *write in terms of* $\cos 2x$

*To integrate* $\tan^2 x\, dx$ *write in terms of* $\sec^2 x$

*To integrate* $\cot^2 x\, dx$ *write in terms of* $\mathrm{cosec}^2 x$

## Definite Integrals

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

$$\text{Area} = \int_a^b f(x)\,dx$$

$$\text{Area} = \int_a^b f(y_1 - y_2)\,dx$$

*see also trapezium rule in numerical methods*

$$\textit{volume rotation} = \pi \int_a^b y^2\,dx$$

## Integrals involving Natural Logs.

*Remember that* $\ln a = {}^{-}\ln({}^{1}/_{a})$.

*In this booklet only the positive log solution is given.*

*Where log solutions are involved, the absolute value should be specified.*

## Notes

† *This follows immediately from* $d(uv) = v\,du + u\,dv$

*For repeated application, differentiate down first column and integrate down second.*

*Answer is the product of the diagonals alternately + then − then + etc.*

## Standard Differentiations and Integrations

| $\int f(x)\,dx = ? + c$ | $f(x)$ | $f'(x) = {}^{dy}/_{dx}$ |
|---|---|---|
| $-\cos\theta$ | $\sin\theta$ | $\cos\theta$ |
| $\sin\theta$ | $\cos\theta$ | $-\sin\theta$ |
| $\ln\lvert\sec\theta\rvert$ | $\tan\theta$ | $\sec^2\theta \equiv 1 + \tan^2\theta$ |
| $\ln\lvert\text{cosec}\,\theta - \cot\theta\rvert$ | $\text{cosec}\,\theta$ | $-\text{cosec}\,\theta \bullet \cot\theta$ |
| $\ln\lvert\sec\theta + \tan\theta\rvert$ † | $\sec\theta$ | $\sec\theta \bullet \tan\theta$ |
| $\ln\lvert\sin\theta\rvert$ | $\cot\theta$ | $-\text{cosec}^2\,\theta$ |
| $x\sin^{-1}x + \sqrt{(1-x^2)}$ | $\sin^{-1}x$ | $^{1}/_{\sqrt{(1-x^2)}}$ |
| $x\cos^{-1}x - \sqrt{(1-x^2)}$ | $\cos^{-1}x$ | $-{}^{1}/_{\sqrt{(1-x^2)}}$ |
| $x\tan^{-1}x - \tfrac{1}{2}\ln\lvert 1 + x^2\rvert$ | $\tan^{-1}x$ | $^{1}/_{(1+x^2)}$ |
| $x\,\text{cosec}^{-1}x + \cosh^{-1}x$ | $\text{cosec}^{-1}x$ | $-{}^{1}/_{(\lvert x\rvert\sqrt{(x^2-1)})}$ |
| $x\sec^{-1}x - \cosh^{-1}x$ | $\sec^{-1}x$ | $^{1}/_{(\lvert x\rvert\sqrt{(x^2-1)})}$ |
| $x\cot^{-1}x + \tfrac{1}{2}\ln\lvert 1 + x^2\rvert$ | $\cot^{-1}x$ | $-{}^{1}/_{(1+x^2)}$ |
| $\cosh x$ | $\sinh x$ | $\cosh x$ |
| $\sinh x$ | $\cosh x$ | $\sinh x$ |
| $\ln\lvert\cosh x\rvert$ | $\tanh x$ | $\text{sech}^2x \equiv 1 - \tanh^2x$ |
| $\ln\lvert\coth x - \text{cosech}\,x\rvert$ | $\text{cosech}\,x$ | $-\text{cosech}\,x \bullet \coth x$ |
| $\tan^{-1}(\sinh x)$ † | $\text{sech}\,x$ | $-\text{sech}\,x \bullet \tanh x$ |
| $\ln\lvert\sinh x\rvert$ | $\coth x$ | $-\text{cosech}^2\,x$ |
| $x\sinh^{-1}x - \sqrt{(x^2+1)}$ | $\sinh^{-1}x$ | $^{1}/_{\sqrt{(x^2+1)}}$ |
| $x\cosh^{-1}x - \sqrt{(x^2-1)}$ | $\cosh^{-1}x$ | $^{1}/_{\sqrt{(x^2-1)}}$ |
| $x\tanh^{-1}x + \tfrac{1}{2}\ln\lvert 1 - x^2)\rvert$ | $\tanh^{-1}x$ | $^{1}/_{(1-x^2)}$ $\quad \lvert x\rvert < 1$ |
| $x\,\text{cosech}^{-1}x + \sinh^{-1}x$ | $\text{cosech}^{-1}x$ | $-{}^{1}/_{(\lvert x\rvert\sqrt{(1+x^2)})}$ |
| $x\,\text{sech}^{-1}x + \sin^{-1}x$ | $\text{sech}^{-1}x$ | $-{}^{1}/_{x\sqrt{(1-x^2)}}$ |
| $x\coth^{-1}x + \tfrac{1}{2}\ln\lvert x^2 - 1\rvert$ | $\coth^{-1}x$ | $^{1}/_{(1-x^2)}$ $\quad \lvert x\rvert > 1$ |

## Notes

† *see notes on the Gudermannian function in the Hyperbolic Booklet*

*the constant of integration is assumed.*



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# Standard Differentiations and Integrations $x \to ax + b$

| $\int f(x)\,dx = ? + c$ | $f(x)$ | $f'(x) = {}^{dy}/_{dx}$ | |
|---|---|---|---|
| $-({}^1/_a)\cos(a\theta + b)$ | $\sin(a\theta + b)$ | $a\cos(a\theta + b)$ | |
| $({}^1/_a)\sin(a\theta + b)$ | $\cos(a\theta + b)$ | $-a\sin(a\theta + b)$ | |
| $({}^1/_a)\ln\lvert\sec(a\theta)\rvert$ | $\tan(a\theta + b)$ | $a\sec^2(a\theta + b)$ | |
| $({}^1/_a)\ln\lvert\mathrm{cosec}(a\theta) - \cot(a\theta)\rvert$ | $\mathrm{cosec}(a\theta + b)$ | $-a\,\mathrm{cosec}(a\theta+b) \bullet \cot(a\theta+b)$ | |
| $({}^1/_a)\ln\lvert\sec(a\theta) + \tan(a\theta)\rvert$ | $\sec(a\theta + b)$ | $a\sec(a\theta + b) \bullet \tan(a\theta+b)$ | |
| $({}^1/_a)\ln\lvert\sin(a\theta)\rvert$ | $\cot(a\theta + b)$ | $-a\,\mathrm{cosec}^2(a\theta + b)$ | |
| $x\sin^{-1}{}^x/_a + \sqrt{(a^2 - x^2)}$ | $\sin^{-1}{}^x/_a$ | ${}^1/_{\sqrt{(a^2 - x^2)}}$ | $x^2 < a^2 \quad a > 0$ |
| $x\cos^{-1}{}^x/_a - \sqrt{(a^2 - x^2)}$ | $\cos^{-1}{}^x/_a$ | $-{}^1/_{\sqrt{(a^2 - x^2)}}$ | |
| $x\tan^{-1}{}^x/_a - ½\,a\ln\lvert a^2+x^2\rvert$ | $\tan^{-1}{}^x/_a$ | ${}^a/_{(x^2 + a^2)}$ | |
| $x\,\mathrm{cosec}^{-1}{}^x/_a + a\cosh^{-1}{}^x/_a$ | $\mathrm{cosec}^{-1}{}^x/_a$ | $-{}^a/_{\lvert x\rvert\sqrt{(x^2 - a^2)}}$ | |
| $x\sec^{-1}{}^x/_a - a\cosh^{-1}{}^x/_a$ | $\sec^{-1}{}^x/_a$ | ${}^a/_{\lvert x\rvert\sqrt{(x^2 - a^2)}}$ | |
| $x\cot^{-1}{}^x/_a + ½\,a\ln\lvert a^2+x^2\rvert$ | $\cot^{-1}{}^x/_a$ | $-{}^a/_{(x^2 + a^2)}$ | |
| $({}^1/_a)\cosh a x$ | $\sinh(a x)$ | $a\cosh(a x)$ | |
| $({}^1/_a)\sinh a x$ | $\cosh(a x)$ | $a\sinh(a x)$ | |
| $({}^1/_a)\ln\lvert\cosh a x\rvert$ | $\tanh(a x)$ | $a\,\mathrm{sech}^2(a x)$ | |
| $({}^1/_a)\ln\lvert\tanh(½ax)\rvert$ | $\mathrm{cosech}(a x)$ | $-a\,\mathrm{cosech}(ax) \bullet \coth(ax)$ | |
| $({}^1/_a)\,2\tan^{-1}(e^{ax})$ | $\mathrm{sech}(a x)$ | $-a\,\mathrm{sech}(ax) \bullet \tanh(ax)$ | |
| $({}^1/_a)\ln(\sinh a x)$ | $\coth(a x)$ | $-a\,\mathrm{cosech}^2(ax)$ | |
| $x\sinh^{-1}{}^x/_a - \sqrt{(x^2 + a^2)}$ | $\sinh^{-1}{}^x/_a$ | ${}^1/_{\sqrt{(x^2 + a^2)}}$ | $a > 0$ |
| $x\cosh^{-1}{}^x/_a - \sqrt{(x^2 - a^2)}$ | $\cosh^{-1}{}^x/_a$ | ${}^1/_{\sqrt{(x^2 - a^2)}}$ | $x > a \quad a > 0$ |
| $x\tanh^{-1}{}^x/_a + ½\,a\ln\lvert a^2 - x^2\rvert$ | $\tanh^{-1}{}^x/_a$ | ${}^a/_{(a^2 - x^2)}$ † | $x^2 < a^2 \quad a > 0$ |
| $x\,\mathrm{cosech}^{-1}{}^x/_a + a\sinh^{-1}{}^x/_a$ | $\mathrm{cosech}^{-1}{}^x/_a$ | $-{}^a/_{x\sqrt{(x^2 + a^2)}}$ | |
| $x\,\mathrm{sech}^{-1}{}^x/_a - a\cos^{-1}{}^x/_a$ | $\mathrm{sech}^{-1}{}^x/_a$ | $-{}^a/_{x\sqrt{(a^2 - x^2)}}$ | |
| $x\coth^{-1}{}^x/_a + ½\ln\lvert x^2 - a^2\rvert$ | $\coth^{-1}{}^x/_a$ | ${}^a/_{(a^2 - x^2)}$ † | $x^2 > a^2$ |

## Notes

† *The differentials appear identical, but remember the functions have no common mapping.*

*Log expressions of integrands can be tidied up by* $\times$ $\div$ *through by any constant.*

# Standard Differentiations and Integrations

| $\int f(x)\,dx = ? + c$ | $f(x)$ | $f'(x) = {}^{dy}/_{dx}$ |
|---|---|---|
| $\tfrac{1}{2} ax^2 + bx$ | $ax + b$ | $a$ |
| ${}^{1}/_{(n+1)} \cdot x^{n+1}$ | $x^{n}$ | $n\,x^{n-1}$ |
| $e^{x}$ | $e^{x}$ | $e^{x}$ |
| $-e^{-x}$ | $e^{-x}$ | $-e^{-x}$ |
| *no general integral.* | $e^{f(x)}$ | $f'(x)\,e^{f(x)}$ |
| $x \ln\lvert x\rvert - x$ | $\ln\lvert x\rvert$ † | ${}^{1}/_{x}$ |
| $({}^{1}/_{\ln w})(x \ln\lvert x\rvert - x)$ | $\log_w \lvert x\rvert$ ‡ | ${}^{1}/_{x.\ln w}$ |
| *no general integral except when* $f(x) = x^n$ | $\ln f(x)$ | ${}^{f'(x)}/_{f(x)}$ |
| $\tfrac{1}{2} x^2 \ln\lvert x\rvert - \tfrac{1}{4} x^2$ | $x \ln\lvert x\rvert$ | $1 + \ln x$ |
| $\ln(\ln\lvert x\rvert)$ | ${}^{1}/_{x \ln \lvert x\rvert}$ | ${}^{-1}/_{x^2 \ln x} - {}^{1}/_{x^2 (\ln x)^2}$ |
| $({}^{1}/_{\ln\lvert w\rvert})w^{x}$ | $w^{x}$ | $w^{x}(\ln w)$ $\quad w>0$ |
| $-({}^{1}/_{\ln\lvert w\rvert})w^{-x}$ | $w^{-x}$ | $-\ln w \times w^{-x}$ |
| *not integrable* | $w^{(1/x)}$ | $-({}^{1}/_{x^2}) \ln w \times w^{(1/x)}$ |
| *not integrable* | $w^{-(1/x)}$ | $({}^{1}/_{x^2}) \ln w \times w^{-(1/x)}$ |
| *not integrable* | $x^{x}$ | $(\ln x + 1) \times x^{x}$ |
| *not integrable* | $x^{-x}$ | $-(\ln x + 1) \times x^{-x}$ |
| *not integrable* | $x^{1/x}$ | $({}^{1}/_{x^2})(1 - \ln x)\, x^{1/x}$ |
| *not integrable* | $x^{-1/x}$ | $({}^{1}/_{x^2})(\ln x - 1)\, x^{-1/x}$ |

## Other Useful Integrals

| | | |
|---|---|---|
| $\sqrt{(x+a)}\ ({}^{2}/_{3})(x + 2a)$ | ${}^{x}/_{\sqrt{(x+a)}}$ | *use substitution* $u = \sqrt{(x+a)}$ |
| $\sqrt{(x-a)}\ ({}^{2}/_{3})(x - 2a)$ | ${}^{x}/_{\sqrt{(x-a)}}$ | *use substitution* $u = \sqrt{(x-a)}$ |
| $\sqrt{(a-x)}\ ({}^{2}/_{3})(2a - x)$ | ${}^{x}/_{\sqrt{(a-x)}}$ | *use substitution* $u = \sqrt{(a-x)}$ |

## Notes

† *Some authorities define* $\ln x$ *to be* $\int_1^x {}^{1}/_{t}\,dt$

‡ *think of this as* ${}^{\ln x}/_{\ln w}$

# Standard Differentiations and Integrations $x \rightarrow ax + b$

| $\int f(x)\,dx = ? + c$ | $f(x)$ | $f'(x) = {}^{dy}/_{dx}$ |
|---|---|---|
| $\frac{1}{2} ax^2 + bx$ | $ax + b$ | $a$ |
| $(\frac{1}{a})(\frac{1}{n+1})(ax+b)^{n+1}$ | $(ax + b)^n$ | $an(ax + b)^{n-1}$ |
| $(\frac{1}{a}) . e^{ax+b}$ | $e^{ax+b}$ | $ae^{ax+b}$ |
| $-(\frac{1}{a}) . e^{-(ax+b)}$ | $e^{-(ax+b)}$ | $-a e^{-(ax+b)}$ |
| *no general form* | $e^{f(x)}$ | $f'(x)\, e^{f(x)}$ |
| $(x+\frac{b}{a}) . \ln\lvert ax+b\rvert - x$ | $\ln\lvert ax + b\rvert$ | $\frac{a}{(ax+b)}$ *and if* $b=0$ *then* $\frac{1}{x}$ |
| $(\frac{1}{\ln w})\{(x+\frac{b}{a}).\ln(ax+b)-x\}$ | $\log_w\lvert ax+b\rvert$ | $(\frac{1}{\ln w})\frac{a}{(ax+b)}$ |
| $\frac{-1}{f(x)}$ | $\frac{f'(x)}{[f(x)]^2}$ | |
| $\frac{1}{2}(x^2-\frac{b^2}{a^2})\ln\lvert ax+b\rvert - \frac{1}{4a}x(ax-2b)$ | $x \ln\lvert ax+b\rvert$ | $\ln\lvert ax+b\rvert + \frac{ax}{ax+b}$ |
| $2\sqrt{f(x)}$ | $\frac{f'(x)}{\sqrt{f(x)}}$ | *used to integrate some inverse functions* |
| $(\frac{1}{a \ln\lvert w\rvert})\, w^{ax+b}$ | $w^{ax+b}$ | $\ln w \bullet a\, w^{ax+b}$ |
| $\frac{-1}{\ln\lvert w\rvert}\, w^{-x}$ | $w^{-(ax+b)}$ | $-\ln w \bullet a\, w^{-(ax+b)}$ |
| *no integral?* | $w^{(1/ax+b)}$ | $-(\frac{1}{(ax+b)^2}) \ln w \bullet a\, w^{(1/ax+b)}$ |
| *no integral?* | $w^{-(1/x)}$ | $(\frac{1}{(ax+b)^2}) \ln w \bullet a\, w^{(-1/ax+b)}$ |
| *no integral?* | $x^{ax+b}$ | $(ax \ln x + ax + b) \bullet x^{ax+b-1}$ |
| *no integral?* | $x^{-(ax+b)}$ | $-(ax \ln x + ax + b) \bullet x^{-(ax+b)-1}$ |
| *no integral?* | $x^{1/ax+b}$ | $(\frac{1}{(ax+b)^2}).(a\, x \ln x - ax - b)\, x^{1/ax+b-1}$ |
| *no integral?* | $x^{-1/ax+b}$ | $(\frac{1}{(ax+b)^2}).(a\, x \ln x - ax - b)\, x^{-1/ax+b-1}$ |

## Some Tricky Integrals (using inverse trig substitutions)

| | | |
|---|---|---|
| $\frac{1}{2}[a^2 \sinh^{-1}\frac{x}{a} + x\sqrt{(x^2 + a^2)}]$ | $\sqrt{(x^2 + a^2)}$ | |
| $\frac{1}{2}[-a^2 \cosh^{-1}\frac{x}{a} + x\sqrt{(x^2 - a^2)}]$ | $\sqrt{(x^2 - a^2)}$ | |
| $\frac{1}{2}[-a^2 \cos^{-1}\frac{x}{a} + x\sqrt{(a^2 - x^2)}]$ | $\sqrt{(a^2 - x^2)}$ | *depending which substitution used* |
| $\frac{1}{2}[a^2 \sin^{-1}\frac{x}{a} + x\sqrt{(a^2 - x^2)}]$ | $\sqrt{(a^2 - x^2)}$ | *will differ by* $\frac{1}{4} a^2 \pi$ |
| $\frac{1}{2}[-a^2 \sinh^{-1}\frac{x}{a} + x\sqrt{(x^2 + a^2)}]$ | $\frac{x^2}{\sqrt{(x^2 + a^2)}}$ | |
| $\frac{1}{2}[a^2 \cosh^{-1}\frac{x}{a} + x\sqrt{(x^2 - a^2)}]$ | $\frac{x^2}{\sqrt{(x^2 - a^2)}}$ | |
| $\frac{1}{2}[-a^2 \cos^{-1}\frac{x}{a} - x\sqrt{(a^2 - x^2)}]$ | $\frac{x^2}{\sqrt{(a^2 - x^2)}}$ | *depending which substitution used* |
| $\frac{1}{2}[a^2 \sin^{-1}\frac{x}{a} - x\sqrt{(a^2 - x^2)}]$ | $\frac{x^2}{\sqrt{(a^2 - x^2)}}$ | *will differ by* $\frac{1}{4} a^2 \pi$ |

# Additional Differentiations and Integrations

| $\int f(\theta)\,d\theta = ? + c$ | $f(\theta)$ | $f'(\theta) = {}^{dy}/_{d\theta}$ |
|---|---|---|
| $\theta/2 - 1/4 \sin(2\theta)$ | $\sin^2\theta$ | $2\sin\theta\cos\theta$ † |
| $\theta/2 + 1/4 \sin(2\theta)$ | $\cos^2\theta$ | $-2\sin\theta\cos\theta$ |
| $\tan\theta - \theta$ | $\tan^2\theta$ | $2\tan\theta\sec^2\theta$ |
| $-\cot\theta$ | $\text{cosec}^2\theta$ | $-2\,\text{cosec}^2\theta\cot\theta$ |
| $\tan\theta$ | $\sec^2\theta$ | $2\sec^2\theta\tan\theta$ |
| $-\cot(2\theta) - \theta$ | $\cot^2\theta$ | $-2\cot\theta\,\text{cosec}^2\theta$ |
| $\sin\theta - \theta\cos\theta$ | $\theta\sin\theta$ | $\theta\cos\theta + \sin\theta$ |
| $\cos\theta + \theta\sin\theta$ | $\theta\cos\theta$ | $\cos\theta - \theta\sin\theta$ |
| *No integral* | $\theta\tan\theta$ | $\theta\sec^2\theta + \tan\theta$ |
| *No integral* | $\theta\,\text{cosec}\,\theta$ | $\text{cosec}\,\theta\,(1 - \theta\cot\theta)$ |
| *No integral* | $\theta\sec\theta$ | $\sec\theta\,(1 + \theta\tan\theta)$ |
| *No integral* | $\theta\cot\theta$ | $-\theta\,\text{cosec}^2\theta + \cot\theta$ |
| $-x/2 + 1/4 \sinh(2x)$ | $\sinh^2 x$ | $2\sinh x\cosh x$ |
| $x/2 + 1/4 \sinh(2x)$ | $\cosh^2 x$ | $2\sinh x\cosh x$ |
| $x - \tanh x\theta$ | $\tanh^2 x$ | $2\tanh x\,(1 - \tanh^2 x)$ |
| $\coth x$ | $\text{cosech}^2 x$ | $-2\,\text{cosech}^2 x\coth x$ |
| $\tanh x$ | $\text{sech}^2 x$ | $-2\,\text{sech}^2 x\tanh x$ |
| $x - \coth x$ | $\coth^2 x$ | $-2\coth x\,\text{cosech}^2 x$ |
| $-\sinh x + x\cosh x$ | $x\sinh x$ | $x\cosh x + \sinh x$ |
| $-\cosh x + x\sinh x$ | $x\cosh x$ | $\cosh x + x\sinh x$ |
| *No integral* | $x\tanh x$ | $x\,\text{sech}^2 x + \tanh x$ |
| *No integral* | $x\,\text{cosech}\,x$ | $\text{cosech}\,x\,(1 - x\coth x)$ |
| *No integral* | $x\,\text{sech}\,x$ | $\text{sech}\,x\,(1 - x\tanh x)$ |
| *No integral* | $x\coth x$ | $-x\,\text{cosech}^2 x + \coth x$ |

## Notes

† = $\sin 2\theta$

## Reduction Formulae *(derived from Integration by Parts)*

Let $I_n = \int \sin^n x \, dx = {}^{-1}/_n \sin^{n-1} x \cdot \cos x + {}^{n-1}/_n \, I_{n-2}$

Let $I_n = \int \cos^n x \, dx = {}^{1}/_n \cos^{n-1} x \cdot \sin x + {}^{(n-1)}/_n \, I_{n-2}$

Let $I_n = \int \tan^n x \, dx = {}^{1}/_{n-1} \tan^{n-1} x - I_{n-2}$

Let $I_n = \int \text{cosec}^n x \, dx = {}^{-1}/_{n-1} \text{cosec}^{n-2} x \cdot \cot x + {}^{n-2}/_{n-1} \, I_{n-2}$

Let $I_n = \int \sec^n x \, dx = {}^{1}/_{n-1} \sec^{n-2} x \cdot \tan x + {}^{n-2}/_{n-1} \, I_{n-2}$

Let $I_n = \int \cot^n x \, dx = {}^{-1}/_{n-1} \cot^{n-1} x - I_{n-2}$

Let $I_n = \int \sinh^n x \, dx = {}^{1}/_n \sinh^{n-1} x \cdot \cosh x - {}^{(n-1)}/_n \, I_{n-2}$

Let $I_n = \int \cosh^n x \, dx = {}^{1}/_n \cosh^{n-1} x \cdot \sinh x - {}^{(n-1)}/_n \, I_{n-2}$

Let $I_n = \int \tanh^n x \, dx = {}^{-1}/_{n-1} \tanh^{n-1} x + I_{n-2}$

Let $I_n = \int \text{cosech}^n x \, dx = {}^{-1}/_{n-1} \text{cosech}^{n-2} x \cdot \coth x - {}^{n-2}/_{n-1} \, I_{n-2}$

Let $I_n = \int \text{sech}^n x \, dx = {}^{1}/_{n-1} \text{sech}^{n-2} x \cdot \tanh x + {}^{n-2}/_{n-1} \, I_{n-2}$

Let $I_n = \int \coth^n x \, dx = {}^{-1}/_{n-1} \coth^{n-1} x + I_{n-2}$

Let $I_n = \int x^n e^x \, dx = x^n e^x - n \, I_{n-1}$

Let $I_n = \int (\ln x)^n \, dx = x (\ln x)^n - n \, I_{n-1}$

## Example layout when using Reduction Formulae

$\int x^3 e^x \, dx = x^3 e^x - 3 \, I_2$

$I_2 = x^2 e^x - 2 \, I_1$

$I_1 = x^1 e^x - 2 \, I_0$

$I_0 = e^x$ *then rebuild step by step*

$I_1 = x^1 e^x - e^x$

$I_2 = x^2 e^x - 2xe^x + 2e^x$

$I_3 = x^3 e^x - 3x^2e^x + 6xe^x - 6e^x$

$= e^x(x^3 - 3x^2 + 6x - 6)$

## Notes

*also note* $\int x^n e^{ax} \, dx = {}^{1}/_a \left(x^n e^{ax} - n \int x^{n-1} e^{ax} \, dx\right)$

## Solution First Order Linear Differential Equations

*Standard form is* $\frac{dy}{dx} + P(x)y = Q(x)$

*The solution is given by first determining the integrating factor given by* $e^{\int P dx}$

*The transformed equation is then* $y \cdot e^{\int P dx} = Q(x) \cdot e^{\int P dx}$

## Bernouilli Differential Equations

*These are of the form*

$$\frac{dy}{dx} + P(x)y = Q(x) \cdot y^n$$

*This is a non linear equation but is transformed to a linear equation by multiplying through by*

$$y^{-n}(1 - n)$$

*This transforms y and produces a modified first order linear differential equation*

## A Few Standard Integrating Factors

$$e^{\int 1 dx} = e^{x}$$

$$e^{\int 2 dx} = e^{2x}$$

$$e^{\int 3 dx} = e^{3x}$$

$e^{\wedge}(\int {}^{1}/_{x}) = x$ $\quad$ $e^{\wedge}(\int {}^{-1}/_{x}) = {}^{1}/_{x}$

$e^{\wedge}(\int {}^{2}/_{x}) = x^2$ $\quad$ $e^{\wedge}(\int {}^{-2}/_{x}) = {}^{1}/_{x^2}$

$e^{\wedge}(\int {}^{3}/_{x}) = x^3$ $\quad$ $e^{\wedge}(\int {}^{-3}/_{x}) = {}^{1}/_{x^3}$

$e^{\wedge}(\int {}^{1}/_{x}) = x$ $\quad$ $e^{\wedge}(\int {}^{-1}/_{x}) = x^{-1}$

$e^{\wedge}(\int {}^{1}/_{2x}) = x^{½}$ $\quad$ $e^{\wedge}(\int {}^{-1}/_{2x}) = x^{-½}$

$e^{\wedge}(\int {}^{1}/_{4x}) = x^{¼}$ $\quad$ $e^{\wedge}(\int {}^{-1}/_{4x}) = x^{-¼}$

$e^{\int \tan x \, dx} = \sec x$ $\quad$ $e^{\int \tanh x \, dx} = \cosh x$

$e^{\int \mathrm{cosec}\, x \, dx} = \mathrm{cosec}\, x - \cot x$ $\quad$ $e^{\int \mathrm{cosech}\, x \, dx} = {}^{1}/_{\mathrm{cosech}\, x + \coth x}$

$e^{\int \sec x \, dx} = \sec x + \tan x$ $\quad$ *doesn't have a neat hyperbolic equivalent*

$e^{\int \cot x \, dx} = \sin x$ $\quad$ $e^{\int \coth x \, dx} = \sinh x$

## Solution Second Order Linear Differential Equations

*if* $a\,{}^{d^2y}/_{dx^2} + b\,{}^{dy}/_{dx} + cy = 0$

*The auxilary quadratic equation is* $am^2 + bm + c = 0$.

*When the auxilary equation has two real roots* $\alpha$ *and* $\beta$ *the general solution is*

$$y = Ae^{\alpha x} + Be^{\beta x}$$

*When the auxilary equation has one coincident root* $\alpha$

$$y = (A + Bx)e^{\alpha x}$$

*When the auxilary equation has the conjugate roots* $p \pm iq$

$$y = e^{px}(A \cos qx + B \sin qx)$$

## Forms of Particular Integral

*if* $a\,{}^{d^2y}/_{dx^2} + b\,{}^{dy}/_{dx} + cy = f(x)$ $\quad y = $ *compl. func. + particular integral*

| | | | | |
|---|---|---|---|---|
| *for* $f(x)$ | = | $k$ | *set* | $y = K$ |
| *for* $f(x)$ | = | $k$ *and* $c = 0$ | *set* | $y = Kx$ |
| *for* $f(x)$ | = | $px$ | *set* | $y = Px + Q$ |
| *for* $f(x)$ | = | $px + q$ | *set* | $y = Px + Q$ |
| *for* $f(x)$ | = | $px^2 + qx + r$ | *set* | $y = Px^2 + Qx + R$ |
| *for* $f(x)$ | = | $ke^{qx}$ | *set* | $y = Ke^{qx}$ *(or try* $Kxe^{qx}$*)* |
| *for* $f(x)$ | = | $\cos kx$ | *set* | $y = P \cos kx + Q \sin kx$ |
| *for* $f(x)$ | = | $\sin kx$ | *set* | $y = P \cos kx + Q \sin kx$ |
| *for* $f(x)$ | = | $\cos kx + \sin kx$ | *set* | $y = P \cos kx + Q \sin kx$ |
| *if* $f(x)$ | = | $\cos k_1x + \sin k_2x$ | *who knows?* | |

## Differentiation of Pcos kx + Qsin kx

$$y = P \cos kx + Q \sin kx$$

$${}^{dy}/_{dx} = -k P \sin kx + k Q \cos kx$$

$${}^{d^2y}/_{dx^2} = -k^2 P \cos kx - k^2 Q \sin kx$$

## Full Solutions of First Order Differential Equations

$$\frac{dy}{dt} = 2t \quad \textit{then} \quad y = t^2 + y_0$$

$$\frac{dy}{dt} = cy \quad \textit{then} \quad y = y_0 e^{ct}$$

$$a\frac{dy}{dt} + by = 0 \quad \textit{then} \quad y = y_0 e^{-(b/a)t}$$

$$a\frac{dy}{dt} + by + c = 0 \quad \textit{then} \quad y = {}^{-c}/_{a} + (y_0 + {}^{c}/_{a})e^{-(b/a)t}$$

$$a\frac{dy}{dt} + by + c = c\sin\omega t \quad \textit{then} \quad y = c\sin(\omega t - \phi) / \sqrt{(a^2 + \omega^2 b^2)}$$

$$\sin\phi = {}^{\omega b}/_{\sqrt{(a^2 + w^2 b^2)}}$$

## Full Solutions of Second Order Differential Equations

$$\frac{d^2y}{dt^2} = a \quad \textit{then} \quad y = \tfrac{1}{2}at^2 + y_1 t + y_0$$

$$\textit{cf} \quad s = ut + \tfrac{1}{2}at^2$$

$$\frac{d^2y}{dt^2} = n^2 y \quad \textit{then} \quad y = \tfrac{1}{2}[(y_0 + {}^{y_1}/_{n})e^{mt} + (y_0 - {}^{y_1}/_{n})e^{-nt}]$$

$$= y_o \cosh nt + {}^{y_1}/_{m} \sinh nt$$

$$\frac{d^2y}{dt^2} = {}^{-}n^2 y \quad \textit{then} \quad y = y_o \cos nt + {}^{y_1}/_{m} \sin nt$$

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \quad \textit{then} \quad y = e^{-mt}(y_o \cosh nt + {}^{y_1+m}/_{n} \sinh nt)$$

$$m = {}^{b}/_{2a}$$

$$n = {}^{\sqrt{(b^2 - 4ac)}}/_{2a}$$

*an alternative representation is* $y = e^{-mt}(Ae^{nt} + Be^{-nt})$

*where* $A = \tfrac{1}{2}[\, y_0 - (\, y_1 + my_0 \,)\,]$ †

*where* $B = \tfrac{1}{2}[\, y_0 (\, y_1 + my_0 \,) - y_0 \,]$ †

*I can also write this equation as a second order system response (this takes me back!)*

$$\frac{d^2y}{dt^2} + 2\xi\omega_n \frac{dy}{dt} + w_n^2 y = 0 \quad \textit{then} \quad y = e^{-\xi\omega t} R \sin(w_n t\sqrt{(1-\xi^2)} + \phi)$$

$$\textit{where} \quad R = \sqrt{\{\, [{}^{(y_1 + y_0)}/_{wn\sqrt{(1-\xi^2)}}]^2 + y_0^2 \,\}}$$

$$\textit{and} \quad \phi = \tan^{-1}\{({}^{1}/_{R})({}^{y_0\omega n\sqrt{(1-\xi^2)}}/_{y_1} + y_0\}$$

## Notes

† *I determined these by taking Laplace transforms and rearranging to an alternative form.*

# Special Integrals

## The Error Function

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

*The probability density function of the Normal Distribution - the constant to make* area = 1

## The Elliptic Integral

$$\int_0^\theta \sqrt{(1 - k^2 \sin^2 \theta)}\, d\theta$$

*Encountered when attempting to calculate length of arc of an ellipse.*

## The Gamma Function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

*For* $n \in Z^+$ *then* $\Gamma(n+1) = n!$ *ie the gamma function is the generalised factorial function.*

## The Beta Function

$$B(n,m) = \int_0^1 x^{n-1}(1-x)^{m-1} dx$$

$$B(n,m) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

*The beta function can itself be used to evaluate integrals such as* $\int_0^1 (1 - x^4)^{-1/2} dx$

## The Bessel Function

$$J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(n\theta - x \sin\theta)\, d\theta$$

*The Bessel function is a function of* $x$ *and* $n$ *and satifies the Bessel equation order* $n$

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0$$

*Bessel functions are important in the advanced study of high-frequency electrical conductors.*

## The Logarithmic Integral

$$\text{Li}(x) = \int_2^x \frac{1}{\ln t} dt$$

*Used in estimates of asymptotic values, including estimate of prime counting function* $\pi(x)$.

## The Cosine Integral

$$\text{Ci}(x) = \int_x^\infty \frac{\cos t}{t} dt$$ *plus its hyperbolic partner*

*Used in many applications including signal processing, semiconductor and quantum physics.*

## The Sine Integral

$$\text{Si}(x) \quad \int_x^\infty \frac{\sin t}{t} dt$$ *plus its hyperbolic partner*

*Used in many applications including signal processing, semiconductor and quantum physics.*

## The Exponential Integral

$$\text{Ei}(x) \quad \int_x^\infty e^{-t} / t\, dt$$

*The integral of any function of the form* $R(x)e^x$, *where* $R$ *is a rational function, reduces to an elementary integral and* $\text{Ei}(x)$

# Mr. G´s Integration Tips

### *Is the function in a standard form?*

eg one of the trig functions,

exponential function, log function, algebraic etc.

### *Is the integrand of the form f´(x) / f(x)?*

f´(x) / f(x) then integral will be of the form ln I f(x) I

### *Is the integrand of the form f´(x) [ f(x) ]$^n$ ?*

Just integrate [ f(x) ]$^n$ , differentiate your answer and adjust.

### *Can the integrand be written in the form $^{ax+b}/_{(x+c)(x+d)}$ ?*

Use partial fractions to simplify into separate fractions and use standard forms to resolve.

### *Is the integrand a product of two distinct functions?*

Use integration by parts $\int u\,dv = uv - \int v\,du$

Choose u for the function that can be easily differentiated and reduced to a constant.

### *Is the integrand a trig function that can be replaced by a standard expression?*

Know your standard trig identities.

### *Can you simplify the integral by changing the variable?*

Simplify the integrand with a substitution.

∞ *rg*

***Counting***

| ***No.*** | ***Greek*** | ***Latin*** |
|---|---|---|
| *1* | *mono* | *uni* |
| *2* | *duo* | *bi* |
| *3* | *tri* | *tri* |
| *4* | *tetra* | *quad* |
| *5* | *penta* | *quin* |
| *6* | *hexa* | *sex* |
| *7* | *hepta* | *sept* |
| *8* | *octo* | *oct* |
| *9* | *nona* | *non* |
| *10* | *deca* | *dec* |

*These booklets are written and produced by Robert Goodhand*

*Although the formulae and expressions given have been individually derived and checked errors do creep in. The booklets are also continuously updated.*

*If you would like the latest issue, just email me at robert.goodhand@gmail.com*