

*Mr. G's little booklet on*

# Logic and Propositional Calculus

*Issue 5.0*

8/16

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## **Mr. G's Little Booklets are**

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## Set Theory

$\in$	=	is an element of	$\notin$	=	is not an element of
$\{x_1, x_2, \dots\}$	=	set with elements $x_1, x_2, \dots$	$\{x \mid x \dots\}$	=	set of all $x$ such that $x \dots$
$\subseteq$	=	is a subset of	$\subset$	=	is a proper subset of
$\cup$	=	union (inclusive OR +)	$\cap$	=	intersection (AND $\times$ )
$\emptyset$	=	the empty set	$\mathcal{E}$ or $\mathcal{U}$	=	the universal set
$n(A)$	=	number elements set $A$	$P(A)$	=	$n(A) / n(\mathcal{E})$

## Number Sets

$\Pi$	=	set of prime numbers	$\{2, 3, 5, \dots\}$
$\mathbb{N}^*$	=	set of counting numbers	$\{1, 2, 3, 4, 5, \dots\}$
$\mathbb{Z}^+$	=	set of positive integers	$\{1, 2, 3, 4, 5, \dots\}$ in practice identical
$\mathbb{N}$	=	set of natural numbers	$\{0, 1, 2, 3, 4, 5, \dots\}$ includes 0
$\mathbb{Z}^-$	=	set of negative integers	$\{\dots, -5, -4, -3, -2, -1\}$
$\mathbb{Z}_n$	=	set of integers modulo $n$	$\{1, 2, 3, \dots, n-1\}$
$\mathbb{E}$	=	set of even integers	$\{\pm 2, \pm 4, \pm 6, \pm 8, \dots\}$
$\mathbb{O}$	=	set of odd integers	$\{\pm 1, \pm 3, \pm 5, \pm 7, \dots\}$
$\mathbb{Z}$	=	set of integers	$\{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \dots\}$ †
$\mathbb{Q}^+$	=	set of +ve rationals	$\{x \mid x \in \mathbb{Q}, x > 0\}$
$\mathbb{Q}_0^+$	=	set of +ve rationals + 0	$\{x \mid x \in \mathbb{Q}, x \geq 0\}$
$\mathbb{Q}$	=	set of rational numbers	$\{p/q, p \in \mathbb{Z}, q \in \mathbb{Z}^+\}$
$\mathbb{P}^+$	=	set of +ve reals	$\{x \mid x \in \mathbb{P}, x > 0\}$
$\mathbb{P}_0^+$	=	set of +ve reals incl. 0	$\{x \mid x \in \mathbb{P}, x \geq 0\}$
$\mathbb{P}$	=	set of real numbers	† $1/0$ is undefined
$\mathbb{X}$	=	set of complex numbers	$\Pi \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{P} \subset \mathbb{X}$
$^*\mathbb{P}$	=	set of hyperreals	‡ includes the infinitesimal $dx$

## Notes

Some mathematicians define  $\mathbb{N}$  as the set natural numbers excluding 0

† Zahlen is German for Number

‡ In the set of hyperreals  $1/dx = \infty$ . Hyperreals formalise infinitesimals in calculus.

## Logic Gates and their Equivalents

p	T	T	F	F		NOT $\neg$	OR $\vee$	NOR $\downarrow$	XOR $\underline{\vee}$	XNOR $\underline{\downarrow}$
0	T	T	T	T	tautology			$\neg(p \downarrow \neg p)$		
1	T	T	T	F	OR		$p \vee q$	$\neg(p \downarrow q)$		
2	T	T	F	T	implication		$p \vee \neg q$	$\neg(p \downarrow \neg q)$		
3	T	T	F	F	p amplifier			$\neg p \downarrow \neg p$		
4	T	F	T	T	implication		$\neg p \vee q$	$\neg(\neg p \downarrow q)$		
5	T	F	T	F	q amplifier			$\neg q \downarrow \neg q$		
6	T	F	F	T	XNOR <sup>†</sup>			#?	$\neg p \underline{\vee} q$	$p \underline{\downarrow} q$
7	T	F	F	F	AND			$\neg p \downarrow \neg q$		
8	F	T	T	T	NAND		$\neg p \vee \neg q$	$\neg(\neg p \downarrow \neg q)$		
9	F	T	T	F	XOR			#?	$p \underline{\vee} q$	$\neg p \underline{\downarrow} q$
10	F	T	F	T	q inverter	$\neg q$		$q \downarrow q$		
11	F	T	F	F	non impl.			$\neg p \downarrow q$		
12	F	F	T	T	p inverter	$\neg p$		$p \downarrow p$		
13	F	F	T	F	non impl.			$p \downarrow \neg q$		
14	F	F	F	T	NOR			$p \downarrow q$		
15	F	F	F	F	denial			$(p \downarrow \neg p)$		

<sup>†</sup>same as  $\leftrightarrow$

Peirce's **XNAND** **XAND**  
 Arrow  $\leftrightarrow$  Equiv.

Don't forget de Morgan's rules.

The **XNAND** gate is functionally identical to the **XOR** gate.

The **XAND** gate is functionally identical to Equivalence.

The **NOR** and **NAND** gates are **functionally complete**.

Machines 6 and 9 require slightly more complex arrangements.

Can you construct the four missing expressions marked '#' ? HINT Draw Venn diagram.

Mr. G investigated **XAND** and **XNAND** in 2001 and defined the symbols  $\underline{\wedge}$  and  $\underline{\uparrow}$

Pointless fact - the computer on Apollo 11 was constructed entirely of **NOR** gates.

AND $\wedge$	NAND $\uparrow$	XAND $\underline{\wedge}$	XNAND $\uparrow$	Implication $\rightarrow$	Contrapos. $\rightarrow$	Non impl. $\setminus$	Contrapos. $\setminus$
	$\neg p \uparrow p$						
	$\neg p \uparrow \neg q$			$\neg p \rightarrow q$	$\neg q \rightarrow p$		
	$\neg p \uparrow q$			$\neg p \rightarrow \neg q$	$q \rightarrow p$		
	$\neg(p \uparrow p)$						
	$p \uparrow \neg q$			$p \rightarrow q$	$\neg q \rightarrow \neg p$		
	$\neg(q \uparrow q)$						
	#?	$p \underline{\wedge} q$	$\neg p \uparrow q$				
$p \wedge q$	$\neg(p \uparrow q)$					$p \setminus \neg q$	$q \setminus \neg p$
	$p \uparrow q$			$p \rightarrow \neg q$	$q \rightarrow \neg p$		
	#?	$\neg p \underline{\wedge} q$	$p \uparrow q$				
	$q \uparrow q$						
$p \wedge \sim q$	$\neg(p \uparrow \neg q)$					$p \setminus q$	$\neg q \setminus \neg p$
	$p \uparrow p$						
$\sim p \wedge q$	$\neg(\neg p \uparrow q)$					$\neg p \setminus \neg q$	$q \setminus p$
$\sim p \wedge \sim q$	$\neg(\neg p \uparrow \neg q)$					$\neg p \setminus q$	$\neg q \setminus p$
	$\neg(\neg p \uparrow p)$						

Sheffer's **XNOR** **XOR**

$\setminus$  diff  $\setminus$  diff

Arrow Non equiv.

*p and q can be exchanged*

Logically  $p \Rightarrow q$  is defined by  $\neg p \vee q$  {the Switcheroo rule - named after Q.Q. Switcheroo}

*p implies q means if you've got p then you must have q but q is independent of p.*

*p only if q*

*rainbows occur only if it's raining*

*q is a necessary condition for p*

*rain is a necessary condition for rainbows*

*p is a sufficient condition for q*

*a rainbow is a sufficient condition for rain*

*q if p*

*it's raining if you have a rainbow*

*Contrapositive combines inverse and converse so logically equivalent to original statement.*

AND $\wedge$	NAND $\uparrow$	XAND $\underline{\wedge}$	XNAND $\updownarrow$	Implication $\rightarrow$	Contrapos. $\rightarrow$	Non impl. $\setminus$	Contrapos. $\setminus$
	$\neg p \uparrow p$						
	$\neg p \uparrow \neg q$			$\neg p \rightarrow q$	$\neg q \rightarrow p$		
	$\neg p \uparrow q$			$\neg p \rightarrow \neg q$	$q \rightarrow p$		
	$\neg(p \uparrow p)$						
	$p \uparrow \neg q$			$p \rightarrow q$	$\neg q \rightarrow \neg p$		
	$\neg(q \uparrow q)$						
	#?	$p \underline{\wedge} q$	$\neg p \updownarrow q$				
$p \wedge q$	$\neg(p \uparrow q)$					$p \setminus \neg q$	$q \setminus \neg p$
	$p \uparrow q$			$p \rightarrow \neg q$	$q \rightarrow \neg p$		
	#?	$\neg p \underline{\wedge} q$	$p \updownarrow q$				
	$q \uparrow q$						
$p \wedge \sim q$	$\neg(p \uparrow \neg q)$					$p \setminus q$	$\neg q \setminus \neg p$
	$p \uparrow p$						
$\sim p \wedge q$	$\neg(\neg p \uparrow q)$					$\neg p \setminus \neg q$	$q \setminus p$
$\sim p \wedge \sim q$	$\neg(\neg p \uparrow \neg q)$					$\neg p \setminus q$	$\neg q \setminus p$
	$\neg(\neg p \uparrow p)$						

Sheffer's **XNOR** **XOR**

$\setminus$  diff  $\setminus$  diff

Arrow

Non equiv.

*p and q can be exchanged*

Logically  $p \Rightarrow q$  is defined by  $\neg p \vee q$  {the Switcheroo rule - named after R.R. Switcheroo}

$p$  implies  $q$  means if you've got  $p$  then you must have  $q$  but  $q$  is independent of  $p$ .

$p$  only if  $q$

*rainbows occur only if it's raining*

$q$  is a necessary condition for  $p$

*rain is a necessary condition for rainbows*

$p$  is a sufficient condition for  $q$

*a rainbow is a sufficient condition for rain*

$q$  if  $p$

*it's raining if you have a rainbow*

Contrapositive combines inverse and converse so logically equivalent to original statement.



## Axioms of Propositional Calculus

$(p \vee p) \Rightarrow p$	$p$ or $p$ implies $p$ - idempotence of the disjunction
$p \Rightarrow (p \vee q)$	$p$ implies $p$ or $q$ - axiom of weakening
$(p \vee q) \Rightarrow (q \vee p)$	$p$ or $q$ implies $q$ or $p$ - commutativity of disjunction
$(p \Rightarrow q) \Rightarrow \{ (r \vee p) \Rightarrow (r \vee q) \}$	$\{ p \text{ implies } q \}$ implies $\{ r \text{ or } p \text{ implies } r \text{ or } q \}$

These four axioms will produce the **tertium non datur** (the excluded middle)

$\langle p \vee \neg p \rangle$  is always true       $p$  or not  $p$  even if  $p$  is indeterminate

## Theorems of Propositional Calculus (after Hofstadter - 2 omitted)

Joining Rule	If $p$ and $q$ are theorems then $(p \wedge q)$ is a theorem.
Separation Rule	If $(p \wedge q)$ is a theorem then both $p$ and $q$ are theorems
Double tilde rule	$\neg\neg$ can be deleted or inserted into any theorem (providing well-formed)
Deduction Rule	If $q$ can be derived when $p$ assumed then $(p \Rightarrow q)$ is a theorem - a fantasy
Carry-over Rule	Inside fantasy, any theorem from the reality one level higher can be used.

## Seven Classical Proofs

$\{ (p \Rightarrow q) \wedge (q \Rightarrow r) \} \Rightarrow (p \Rightarrow r)$	Proof by hypothetical syllogism- theory consequences If $p$ implies $q$ and $q$ implies $r$ then $p$ implies $r$
$(p \vee q) \wedge \neg q \Rightarrow p$	Proof by disjunctive syllogism A - by denying affirms If either $p$ or $q$ true but $q$ false then $p$ must be true
$\neg(p \vee q) \wedge \neg q \Rightarrow p$	Proof by disjunctive syllogism B - by affirming denies If (not $p$ and not $q$ ) and $p$ then not $q$ must be true.
nb Using de Morgan Rule I found it clearer to express $\neg(p \vee q)$ as (not $p$ and not $q$ ).	
$\{ (p \Rightarrow q) \wedge p \} \Rightarrow q$	Proof by detachment - by affirming affirms If $p$ then $q$ and $p$ is true then $q$ must be true.
$\{ (p \Rightarrow q) \wedge \neg q \} \Rightarrow \neg p$	Proof by indirect reasoning - by denying denies If $p$ then $q$ but if $q$ is false then $p$ must also be false.
$(\neg p \Rightarrow q) \wedge (\neg p \Rightarrow \neg q) \Rightarrow p$	Proof by contradiction - reductio ad absurdum If denying $p$ leads to a contradiction then $p$ must be true.
$(p \vee q) \wedge \{ (p \Rightarrow r) \wedge (q \Rightarrow r) \} \Rightarrow r$	Proof by cases. If either $p$ or $q$ is true and either imply $r$ then $r$ must also be true.
also $\{ (p \vee q) \wedge (p \vee \neg q) \} \Rightarrow p$	this is an expanded version of disjunctive syllogism If it's $p$ or $q$ and $p$ or not $q$ then $p$ must be true



Greek Alphabet			Principle/Simplest Use	English	Type	
alpha	A	<i>not used</i>	$\alpha$	<i>first root of quadratic</i>	a	<b>a</b>
beta	B	<i>Beta function</i>	$\beta$	<i>second root of quadratic</i>	b	<b>b</b>
gamma	$\Gamma$	<i>Gamma function</i>	$\gamma$	<i>Euler's constant</i>	g	<b>g</b>
delta	$\Delta$	<i>Difference operator</i>	$\delta$	<i>small increment</i>	d	<b>d</b>
epsilon	E	<i>not used</i>	$\epsilon$	<i>error</i>	short e	<b>e</b>
zeta	Z	<i>not used</i>	$\zeta$	<i>Riemann zeta function</i>	z	<b>z</b>
eta	H	<i>not used</i>	$\eta$	<i>efficiency</i>	long e	<b>h</b>
theta	$\Theta$	<i>asymp. tight bound</i>	$\theta$	<i>angle</i>	th	<b>q</b>
iota	I	<i>not used</i>	$\iota$	<i>imaginary unit</i>	i	<b>i</b>
kappa	K	<i>not used</i>	$\kappa$	<i>curvature</i>	k	<b>k</b>
lambda	$\Lambda$	<i>diag. matrix eigen-values</i>	$\lambda$	<i>failure rate</i>	l	<b>l</b>
mu	M	<i>not used</i>	$\mu$	<i>population mean</i>	m	<b>m</b>
nu	N	<i>not used</i>	$\nu$	<i>poisson ratio</i>	n	<b>n</b>
xi	$\Xi$	<i>grand canonical ensemble</i>	$\xi$	<i>damping coefficient</i>	x	<b>x</b>
omicron	O	<i>limiting behaviour function</i>	$\omicron$	<i>generally not used</i>	short o	<b>o</b>
pi	$\Pi$	<i>Product operator</i>	$\pi$	<i>ratio <math>c/d</math> circle</i>	p	<b>p</b>
rho	P	<i>not used</i>	$\rho$	<i>correlation coefficient</i>	r	<b>r</b>
sigma	$\Sigma$	<i>summation</i>	$\sigma$	<i>standard deviation</i>	s	<b>s</b>
tau	T	<i>not used</i>	$\tau$	<i>mean lifetime</i>	t	<b>t</b>
upsilon	Y	<i>Bessel function</i>	$\upsilon$	<i>generally not used</i>	u	<b>u</b>
phi	$\Phi$	<i>cumulative function</i>	$\phi$	<i>golden ratio</i>	ph	<b>f</b>
phi (alt.)	$\varphi$	<i>not used</i>	$\varphi$	<i>normal function</i> <i>scalar potential</i>	ph	<b>j</b>
chi	X	<i>probability function</i>	$\chi^2$	<i>chi-squared prob.function</i>	ch	<b>c</b>
psi	$\Psi$	<i>not used</i>	$\psi$	<i>wave function</i>	ps	<b>y</b>
omega	$\Omega$	<i>mathematical constant</i>	$\omega$	<i>angular frequency</i>	long o	<b>w</b>
stigma	$\varsigma$					<b>v</b>
pomega			$\varpi$	<i>angular velocity</i>		<b>v</b>

## Orders of Magnitude

septillionth	yocto-	y	$10^{-24}$	septillion	yotta-	Y	$10^{24}$
sextillionth	zepto-	z	$10^{-21}$	sextillion	zetta-	Z	$10^{21}$
quintillionth	atto-	a	$10^{-18}$	quintillion	exa-	E	$10^{18}$
quadrillionth	femto-	f	$10^{-15}$	quadrillion	peta-	P	$10^{15}$
trillionth	pico-	p	$10^{-12}$	trillion	tera-	T	$10^{12}$
billionth	nano-	n	$10^{-9}$	billion	giga-	G	$10^9$
millionth	micro-	$\mu$	$10^{-6}$	million	mega-	M	$10^6$
thousandth	milli-	m	$10^{-3}$	thousand	kilo-	k	$10^3$
hundredth	centi-	c	$10^{-2}$	hundred	hecto-	h	$10^2$
tenth	deci-	d	$10^{-1}$	ten	deca-	da	$10^1$
one	-	-	$10^0$	one	-	-	$10^0$

## Mathematical Constants - 30 decimals (last place not rounded)

<i>pi</i>	$\pi$	=	3.14159 26535 89793 23846 26433 83279...
<i>exponential</i>	e	=	2.71828 18284 59045 23536 02874 71352...
<i>Pythagoras's</i>	$\sqrt{2}$	=	1.41421 35623 73095 04880 16887 24209...
	$\sqrt{3}$	=	1.73205 08075 68877 29352 74463 41505...
	log 2	=	0.69314 71805 59945 30941 72321 21458...
<i>golden ratio</i>	$\phi$	=	1.61803 39887 49894 84820 45868 34365...
<i>Euler-Mascheroni</i>	$\gamma$	=	0.57721 56649 01532 86060 65120 90082...
<i>Feigenbaum's</i>	$\delta$	=	4.66920 16091 02990 67185 32038 20466...
	$\xi(2)$	=	1.64493 40668 48226 43647 24151 66646...
<i>Apery's</i>	$\xi(3)$	=	1.20205 69031 59594 28539 97381 61511...
	$\xi(4)$	=	1.08232 32337 11138 19151 60036 96541...
<i>Euler's</i>	$\xi(5)$	=	1.03692 77551 43369 92633 13654 86457...
	$\xi(6)$	=	1.01734 30619 84449 13971 45179 29790...
	$e^\pi$	=	23.14069 26327 79269 00572 90863 67948...

## Prime Numbers (in columns of 25)

2	101	233	383	547	701	877	1049	1229	1429	1597	1783
3	103	239	389	557	709	881	1051	1231	1433	1601	1787
5	107	241	397	563	719	883	1061	1237	1439	1607	1789
7	109	251	401	569	727	887	1063	1249	1447	1609	1801
11	113	257	409	571	733	907	1069	1259	1451	1613	1811
13	127	263	419	577	739	911	1087	1277	1453	1619	1823
17	131	269	421	587	743	919	1091	1279	1459	1621	1831
19	137	271	431	593	751	929	1093	1283	1471	1627	1847
23	139	277	433	599	757	937	1097	1289	1481	1637	1861
29	149	281	439	601	761	941	1103	1291	1483	1657	1867
31	151	283	443	607	769	947	1109	1297	1487	1663	1871
37	157	293	449	613	773	953	1117	1301	1489	1667	1873
41	163	307	457	617	787	967	1123	1303	1493	1669	1877
43	167	311	461	619	797	971	1129	1307	1499	1693	1879
47	173	313	463	631	809	977	1151	1319	1511	1697	1889
53	179	317	467	641	811	983	1153	1321	1523	1699	1901
59	181	331	479	643	821	991	1163	1327	1531	1709	1907
61	191	337	487	647	823	991	1171	1361	1543	1721	1913
67	193	347	491	653	827	1009	1181	1367	1549	1723	1831
71	197	349	499	659	829	1013	1187	1373	1553	1733	1933
73	199	353	503	661	839	1019	1193	1381	1559	1741	1949
79	211	359	509	673	853	1021	1201	1399	1567	1747	1951
83	223	367	521	677	857	1031	1213	1409	1571	1753	1973
89	227	373	523	683	859	1033	1217	1423	1579	1759	1979
97	229	379	541	691	863	1039	1223	1427	1583	1777	1999

## Notes

Prime Number Theorem states that the number of primes up to  $n$ ,  $\pi_n \sim n / \ln(n)$

Alternatively the  $n^{\text{th}}$  prime number  $p_n \sim n \ln(n)$ . So  $p_{300} \sim 300 \ln 300 = 1711$  (cf 1999)

If  $\text{li} = \int_{\text{Int}}^{\text{dt}}$  then  $\text{Li}(x) = \int_2^x \text{dt} / \text{Int} = \text{li}(x) - \text{li}(2)$  is a better approximation to  $\pi(x)$

Goodhand's conjecture states the percent proportion of primes approximately equals the percent that  $n / \ln(n)$  underestimates  $p(n)$ . Hence  $\pi(n) \text{ better } \approx \frac{1}{2} (1 - \sqrt{1 - \frac{4}{\ln(n)}})$

## **Counting**

<b>No.</b>	<b>Greek</b>	<b>Latin</b>
1	mono	uni
2	duo	bi
3	tri	tri
4	tetra	quad
5	penta	quin
6	hexa	sex
7	hepta	sept
8	octo	oct
9	nona	non
10	deca	dec

*These booklets are written and produced by Robert Goodhand*

*Although the formulae and expressions given have been individually derived and checked errors do creep in. The booklets are also continuously updated.*

*If you would like the latest issue, just email me at [robert.goodhand@gmail.com](mailto:robert.goodhand@gmail.com)*