*Mr. G's little booklet on*

# Vectors and Matrices

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*rg*

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# Vectors 1 - Linear Transformations

*A transformation is linear if*

$$T(a_1\underline{\mathbf{v}}_1 + a_2\underline{\mathbf{v}}_2) \quad = \quad a_1T(\underline{\mathbf{v}}_1) + a_2T(\underline{\mathbf{v}}_2)$$

*where* $T:\underline{\mathbf{v}}\ \alpha T(\underline{\mathbf{v}})$

*The set of vectors being transformed are termed the* **domain** *and the new set the* **image**.

# Inverse Transformations

*If T is a one-to-one transform then the inverse linear transform* $T^{-1}$ *exists.*

*If* $T\,T^{-1} = T^{-1}\,T = I$

# Laws of Vector Algebra

## Commutative Law

| | | | |
|---|---|---|---|
| $\underline{\mathbf{v}} + \underline{\mathbf{w}}$ | $=$ | $\underline{\mathbf{w}} + \underline{\mathbf{v}}$ | *Additive* |

## Associative Law

| | | | |
|---|---|---|---|
| $\underline{\mathbf{v}} + (\underline{\mathbf{w}} + \underline{\mathbf{x}})$ | $=$ | $(\underline{\mathbf{v}} + \underline{\mathbf{w}}) + \underline{\mathbf{x}}$ | |
| $m(n\underline{\mathbf{v}})$ | $=$ | $mn(\underline{\mathbf{v}})$ | |
| | $=$ | $n(m\underline{\mathbf{v}})$ | *Multiplicative* |

## Distributive Law

| | | | |
|---|---|---|---|
| $(m+n)\underline{\mathbf{v}}$ | $=$ | $m\underline{\mathbf{v}} + n\underline{\mathbf{v}}$ | *Additive* |
| $m(\underline{\mathbf{v}} + \underline{\mathbf{w}})$ | | $m\underline{\mathbf{v}} + m\underline{\mathbf{w}}$ | *Multiplicative* |

# Notes

*Neither vectors nor matrices obey the product commutative law.*

*Operations that do obey are more the exception than the rule.*

*eg The "son of Harry's mother" is a different person to the "mother of Harry's son".*

## Vectors 2 Magnitude and Modulus

*Define the vector* $\underline{v}$ *by the matrix* $[v1 \ \ v2 \ \ v3]$

$$|\underline{v}| = \sqrt{(v_1^2 + v_2^2 + v_3^2)}$$

$$|\underline{v} + \underline{w}|^2 = |\underline{v}|^2 + |\underline{w}|^2$$

*norm. (unit) vector or "versor"* $\hat{\underline{u}} = \underline{u} / |u|$ *pronounced "u hat"*

## Unit Vectors

$$\underline{v} = x\underline{i} + y\underline{j} + z\underline{k}$$

$\underline{i}.\underline{i} = \underline{j}.\underline{j} = \underline{k}.\underline{k} = 1$ *from scalar product rule*

$\underline{i} \bullet \underline{j} = \underline{k}$ **and** $\underline{j} \bullet \underline{k} = \underline{i}$ **and** $\underline{k} \bullet \underline{i} = \underline{j}$ *from scalar product rule*

*Hence by multiplying out* $\underline{a} \bullet \underline{b} = a_1b_1 + a_2b_2 + a_3b_3$

## Scalar (dot •) Product

$\underline{v} \bullet \underline{w} = |\underline{v}| \, |\underline{w}| \cos\theta$ *this is a scalar quantity*

*by x out in unit vector form* $\underline{v} \bullet \underline{w} = v_1w_1 + v_2w_2 + v_3w_3$

$$\cos\theta = v_1w_1 + v_2w_2 + v_3w_3 / |\underline{v}| \, |\underline{w}|$$

*if v and w parallel* $\underline{v} \bullet \underline{w} = x\underline{i} + y\underline{j} + z\underline{k}$

*if v and w perpendicular* $\underline{v} \bullet \underline{w} = 0$

*resolved part* $\underline{v}$ *in direction* $\underline{w}$ $= \underline{v}\,\underline{w} / |w|$

## Vector (cross) Product

$\underline{v} \times \underline{w} = |\underline{v}| \, |\underline{w}| \sin\theta \, \hat{\underline{u}}$ *this is a vector quantity*

$\hat{\underline{u}}$ *is the unit vector perpendicular to both* $\underline{v}$ *and* $\underline{w}$ *(right hand corkscrew rule)*

*hence* $\hat{\underline{u}} = \underline{u} / |u|$

$|\underline{v} \times \underline{w}| = |\underline{v}| \, |\underline{w}| \sin\theta$ *this is a scalar quantity*

$$\underline{v} \times \underline{w} = i \begin{vmatrix} v2 & v3 \\ w2 & w3 \end{vmatrix} - j \begin{vmatrix} v1 & v3 \\ w1 & w3 \end{vmatrix} + k \begin{vmatrix} v1 & v2 \\ w1 & w2 \end{vmatrix}$$

*cross products are not commutative :* $\mathbf{v} \times \mathbf{w} = {}^{-}\mathbf{w} \times \mathbf{v}$

*the vector product gives a vector perpendicular to* $\underline{v}$ *and* $\underline{w}$ *obeying the right hand rule*

*as an example of application* $A_\Delta = \frac{1}{2} |\underline{v} \times \underline{w}|$

# Vectors 3 - Equation Line

*A point with position vector* $\underline{\mathbf{a}} = a_1 \underline{\mathbf{i}} + a_2 \underline{\mathbf{j}} + a_3 \underline{\mathbf{k}}$

*and let direction vector be* $\underline{\mathbf{v}} = v_1 \underline{\mathbf{i}} + v_2 \underline{\mathbf{j}} + v_3 \underline{\mathbf{k}}$

$\underline{\mathbf{r}} = \underline{\mathbf{a}} + \lambda \underline{\mathbf{b}}$ where $\underline{\mathbf{b}} = \underline{\mathbf{v}} - \underline{\mathbf{a}}$

## Parametric Form

$$x = x_0 + \lambda \ell$$
$$y = y_0 + \lambda m$$
$$z = z_0 + \lambda n$$

## Cartesian Equation

$$(x - x_0) /_{\ell} = (y - y_0) /_{m} = (z - z_0) /_{n}$$

## Vector Equation Plane

$\underline{\mathbf{a}}$ *is any point on the plane*

$\mathbf{r} = \underline{\mathbf{a}} + \lambda \underline{\mathbf{u}} + \mu \underline{\mathbf{v}}$ *not a lot of use*

*let* $\underline{\mathbf{n}} = \underline{\mathbf{u}} \times \underline{\mathbf{v}}$ *be a normal to the plane*

$\mathbf{r} \bullet \mathbf{n} = \mathbf{a} \bullet \mathbf{n}$

$\underline{\mathbf{r}} - \underline{\mathbf{a}}$ *is a vector in the plane*

*and the two are perpendicular*

## Cartesian Equation Plane

$ax + by + cz + d = 0$ $d = {}^{-}\underline{\mathbf{a}} \bullet \underline{\mathbf{n}}$

## Notes

## Determinants

*Value associated with square matrix that determines if a set of linear equations have a solution.*

*The determinant of square matrix* **A** *is written* $|\mathbf{A}|$

*The determinant of the Identity matrix* $|\mathbf{I}_n| = 1$ *(one)*

*The determinant of the tranpose matrix* $|\mathbf{A}^T| = |\mathbf{A}|$ *(the original matrix)*

*The determinant of the inverse matrix* $|\mathbf{A}^{-1}| = {}^1/_{|\mathbf{A}|}$

$|\mathbf{AB}| = |\mathbf{A}|\,|\mathbf{B}|$ *(determinants are distributive in multiplication)*

$|c\mathbf{A}| = c^n|\mathbf{A}|$ *( for an n × n matrix)*

$|\bar{\mathbf{A}}| = \overline{|\mathbf{A}|}$ *(the det. of a complex conjugate = complex conjugate of the det..)*

*The* **determinant** *of square matrix* **det A** *or* $|\mathbf{A}|$ *is a scale factor under linear transform.*

$$|\mathbf{A}| = ad - bc \qquad \text{for } \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

*A* **minor** $M_{ij}$ *of matrix* **A** *is determinant of sub matrix after removing* $i^{th}$ *row &* $j^{th}$ *column.*

## Notes on Determinants (quick fixes to calculate the determinant)

*Switching two rows or two columns changes the sign of the determinant.*

*Scalars can be factored out.*

*Adding or subtracting a row (or column) to another row (or column) doesn't change determinant.*

*Hence multiples of rows (or columns) can be added to other rows (or columns).*

*Swapping row with columns doesn't change the determinant.*

*A determinant with a zero row or column has value 0.*

*If any two rows (or columns) are identical or in proportion the determinant is zero.*

*(this indicates we have an* **underspecified system,** *so inverse matrix does not exist)*

## Matrices

*A matrix is an array of numbers with n rows and m columns.*

$\mathbf{A} \pm \mathbf{B} = \mathbf{B} \pm \mathbf{A}$ — **A** *and* **B** *must be of the same order*

$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$

$\mathbf{A}_{(n\times m)} \times \mathbf{B}_{(m\times p)} = \mathbf{C}_{(n\times p)}$ — *rows × columns*

$\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}$ — *assumming the multiplication actually exists.*

*A half of* $\mathbf{A} = \tfrac{1}{2}\mathbf{A}$ — *nb.* $^{\mathbf{A}}/_{2}$ *is undefined while* $\tfrac{1}{2}\mathbf{A}$ *is a scalar multiplication.*

*The identity matrix is termed* **I**. *It is a square matrix with diagonal "1" s.*

$\mathbf{A} \times \mathbf{I} = \mathbf{A}$ $\qquad$ $\mathbf{I} \times \mathbf{A} = \mathbf{A}$

**I** *has 7 colleagues which rg writes* $\mathbf{I}_{ref.x}$ $\mathbf{I}_{ref.y}$ $\mathbf{I}_{rot90}$ $\mathbf{I}_{rot\,-90}$ $\mathbf{I}_{rot180}$ $\mathbf{I}_{ref.y=x}$ $\mathbf{I}_{ref.y=-x}$

*The* **determinant** *of square matrix* **det A** *or* $|\mathbf{A}|$ *is a scale factor under linear transform.*

$\det \mathbf{A} = ad - bc$ $\qquad$ for $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

*A* **minor** $M_{ij}$ *of matrix* **A** *is determinant of sub matrix after removing* $i^{th}$ *row &* $j^{th}$ *column.*

*A* **cofactor** *entry* $C_{ij}$ *is a "signed minor". A matrix of cofactors is termed the* **conjugate**

*The* **transpose** *of a matrix* $\mathbf{A}^T$ *is the matrix formed where* $\mathbf{A}_{ij}$ *becomes* $\mathbf{A}_{ji}$

$(\mathbf{A} \times \mathbf{B})^T = \mathbf{B}^T \times \mathbf{A}^T$ — *Note the exchange of* **A** *and* **B**

*The* **transpose** *of a matrix of cofactors is termed the* **adjugate**.

*The adjugate matrix divided by scalar quantity* det **A** *gives the inverse* $\mathbf{A}^{-1}$

$\mathbf{A} \times \mathbf{A}^{-1} = \mathbf{I}$ — $\mathbf{A}^{-1}$ *is the inverse of* **A** *which must be square matrix.*

$(\mathbf{A} \times \mathbf{B})^{-1} = \mathbf{B}^{-1} \times \mathbf{A}^{-1}$ — *Note the exchange of* **A** *and* **B**

*The inverse exists if* det **A** *is non singular* $\det \mathbf{A} \neq 0$

*Assume* **B** *is unknown and* **A** *and* **C** *known where* $\mathbf{A} \times \mathbf{B} = \mathbf{C}$

*then* $\mathbf{B} = \mathbf{A}^{-1} \times \mathbf{C}$ — *and rhs can be fully determined.*

*A matrix can have complex elements.*

*The conjugate transpose of such a matrix is termed the adjoint* $\mathbf{A}^*$ **or** $\mathbf{A}^\dagger = (\overline{\mathbf{A}})^T = \overline{\mathbf{A}^T}$ .

## Notes on Determinants (quick fixes to calculate the determinant)

*Adding or subtracting row or column to another row or column doesn't change determinant.*

*Swapping row with columns doesn't change the determinant.*

*If any two rows or columns are identical or in proportion the determinant is zero.*

*(this indicates we have an* **underspecified system,** *so inverse matrix does not exist)*

# Matrix Multiplication

| | A | | | B | | | | A x B | | Transform |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | *point p* | *point q* | | | | | |
| $I$ | 1 | 0 | × | 1 | 2 | *x coord* | = | 1 | 2 | *Identity* |
| | 0 | 1 | | 3 | 4 | *y coord* | | 3 | 4 | *(enlargement)* |
| $I_{ref.y}$ | -1 | 0 | × | 1 | 2 | *x coord* | = | -1 | -2 | *Reflection* |
| | 0 | 1 | | 3 | 4 | *y coord* | | 3 | 4 | *in y-axis* |
| $I_{ref.x}$ | 1 | 0 | × | 1 | 2 | *x coord* | = | 1 | 2 | *Reflection* |
| | 0 | -1 | | 3 | 4 | *y coord* | | -3 | -4 | *in x-axis* |
| $I_{rot.180}$ | -1 | 0 | × | 1 | 2 | *x coord* | = | -1 | -2 | *180° rotation* |
| | 0 | -1 | | 3 | 4 | *y coord* | | -3 | -4 | *enlarge by* $^{-}1$ |
| $I_{ref.y=x}$ | 0 | 1 | × | 1 | 2 | *x coord* | = | 3 | 4 | *Reflection* |
| | 1 | 0 | | 3 | 4 | *y coord* | | 1 | 2 | *in line* $y = x$ |
| $I_{rot.-90}$ | 0 | -1 | × | 1 | 2 | *x coord* | = | -3 | -4 | *90° rotation* |
| | 1 | 0 | | 3 | 4 | *y coord* | | 1 | 2 | *anti-clockwise* |
| $I_{rot.90}$ | 0 | 1 | × | 1 | 2 | *x coord* | = | 3 | 4 | *90° rotation* |
| | -1 | 0 | | 3 | 4 | *y coord* | | -1 | -2 | *clockwise* |
| $I_{ref.y=-x}$ | 0 | -1 | × | 1 | 2 | *x coord* | = | -3 | -4 | *Reflection* |
| | -1 | 0 | | 3 | 4 | *y coord* | | -1 | -2 | *in line* $y = {}^{-}x$ |

# Notes

*Note transforming matrix × the matrix to be transformed - "other way round" isn't consistent.*

*The order of operations for functions and transformations is always right to left.*

## Transformations

*let* $v \longrightarrow Mv$ *where v is the appropriate column matrix*

*Rotation through* $\theta$ *about origin*

$$M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$ *and from this we immediately derive*

$$\begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \times \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

*and multiplying out the two matrices will give us the circular addition formulae*

*Reflection in* $y = x \tan\theta$

$$M = \begin{bmatrix} \cos2\theta & \sin2\theta \\ \sin2\theta & -\cos2\theta \end{bmatrix}$$

## Representation Complex Numbers by Matrices

*let* $z = a + ib$ *be represented by* $a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ †

*Hence* $z = a + ib$

$$= \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$ ‡

*So* $z^* = a - ib$

$$= \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

$$= \begin{bmatrix} a & -b \\ b & a \end{bmatrix}^T$$ *hence conjugate represented by the transpose matrix*

*Further* $|z|^2 = a^2 + b^2$

$$= \begin{vmatrix} a & -b \\ b & a \end{vmatrix}$$

*so the modulus* ² *is represented by the determinant*

*Finally we can calculate* $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}^{-1} = {}^{1}/_{a^2+b^2} \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

$$= {}^{a - ib}/_{a^2 + b^2}$$

$$= z^{-1}$$ *as expected*

## Notes

† *These are the matrices* $\mathbf{I}$ *and* $\mathbf{I}_{rot-90}$

‡ *The Jacobian matrix satisfying "Cauchy-Riemann" takes this form for mappings* $P^2 \rightarrow P^2$.

*The elements of matrices can be complex numbers or even matrices.*

# Practical Example of Matrices

**Quantity Matrix** | **Price Matrix**

| Bread | Milk | Eggs |
|---|---|---|
| 2 | 4 | 3 |

| | Tesco |
|---|---|
| Bread | 120 |
| Milk | 75 |
| Eggs | 180 |

| | Tesco |
|---|---|
| Total bill | 1080 |

**Quantity Matrix** | **Price Matrix**

| Bread | Milk | Eggs |
|---|---|---|
| 2 | 4 | 3 |

| | Tesco | Asda |
|---|---|---|
| Bread | 120 | 110 |
| Milk | 75 | 70 |
| Eggs | 180 | 190 |

| | Tesco | Asda |
|---|---|---|
| Total bill | 1080 | 1070 |

**Quantity Matrix** | **Price Matrix**

| | Bread | Milk | Eggs |
|---|---|---|---|
| Charlie | 2 | 4 | 3 |
| Nicole | 3 | 2 | 5 |

| | Tesco | Asda |
|---|---|---|
| Bread | 120 | 110 |
| Milk | 75 | 70 |
| Eggs | 180 | 190 |

| Tesco | Asda | |
|---|---|---|
| 1080 | 1070 | Charlie |
| 1410 | 1420 | Nicole |

*But if they choose a combined shop it doesn't matter which.* 2490 2490

# Notes

## Transformations - Combined Transformations

*If point P, coord ($P_x$,$P_y$) transforms to Q ($Q_x$,$Q_y$) then the matrix relationship is*

**T** × **P** = **Q** *where* **T** *is a 2x2 matrix, and* **P** *and* **Q** *are 2x1 matrices (the coordinates)*

*Such* ***affine*** *transformations (preserve straight lines and areas) can be extended to P Q R…*

*As all rigid body movements are rotations, translations or a combination of the two.*

*and setting aside a detailed matrix analysis, there are just two restrictions to consider*

*1) Chiral shapes are not identical to their mirror images.*

*2) Translations cannot be represented by a 2x2 matrix*

*These restrict which combined transformations can be represented by a single transform.*

*In general* $\mathbf{T_1 \times T_2 \neq T_2 \times T_1}$ *unless otherwise stated.*

## A Mapping of Combined Transforms (rotation, reflection, translation)

*Studying three transforms gives 9 combinations plus 2 more covering ambiguous outcomes.*

| **1st Transform** | *followed by* | **2nd Transform** | *equivalent to* | **Single Transform** |
|---|---|---|---|---|
| Rotation | + | Rotation | ≡ | Rotation |

*where centre* $Rot_s$ *= mean of centres* $Rot_1$ *and* $Rot_2$ *and* $Rot_{s\,\theta} = Rot_{1\,\theta} + Rot_{2\,\theta}$

*order irrelevant.*

| | | | | |
|---|---|---|---|---|
| Rotation | + | Rotation | ≡ | Translation |

*where* $Rot_{1\,\theta} + Rot_{2\,\theta} = 0$ *(no net rotation)*

| | | | | |
|---|---|---|---|---|
| Rotation | + | Translation | ≡ | Rotation |
| Translation | + | Rotation | ≡ | Rotation |

*standard construction identifies centre rotation two similar chiral shapes.* †

| | | | | |
|---|---|---|---|---|
| Rotation | + | Reflection | ≡ | Reflection |
| Reflection | + | Rotation | ≡ | Reflection |
| Reflection | + | Reflection | ≡ | Rotation |

***if and only if*** *the centre of rotation is at the intersect of the two lines of reflection.* ‡

*The order of the transforms affects the degree of rotation.*

| | | | | |
|---|---|---|---|---|
| Translation | + | Translation | ≡ | Translation |

*Fairly obviously the combined vector translation is the vector sum of the two. Order irrelevant.*

| | | | | |
|---|---|---|---|---|
| Reflection | + | Reflection | ≡ | Translation |

*where the reflection lines are parallel (centre of rotation at infinity)*

| | | | | |
|---|---|---|---|---|
| Translation | + | Reflection | ≡ | *does not exist* |
| Reflection | + | Translation | ≡ | *does not exist* |

*because one reflection changes the chirality of the object. Reflections must be "undone".*

## Notes

† *Although* $\mathbf{T_1 \times T_2 \neq T_2 \times T_1}$ *the final vector inverts and there is a sign change.*

‡ *This is a key result because students are often asked to find the "second" transform.*

***Counting***

| ***No.*** | ***Greek*** | ***Latin*** |
|---|---|---|
| *1* | *mono* | *uni* |
| *2* | *duo* | *bi* |
| *3* | *tri* | *tri* |
| *4* | *tetra* | *quad* |
| *5* | *penta* | *quin* |
| *6* | *hexa* | *sex* |
| *7* | *hepta* | *sept* |
| *8* | *octo* | *oct* |
| *9* | *nona* | *non* |
| *10* | *deca* | *dec* |

*These booklets are written and produced by Robert Goodhand*

*Although the formulae and expressions given have been individually derived and checked errors do creep in. The booklets are also continuously updated.*

*If you would like the latest issue, just email me at robert.goodhand@gmail.com*