Mr. G's little booklet on

Miscellaneous Aspects of Mathematics

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Mr. G's Little Booklets are

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Indices

$$a^m \bullet a^n = a^{m+n}$$

$$a^m / a^n = a^{m-n}$$

$$a^{-m} = 1/a^m$$

$$a^{1/m} = \sqrt[m]{a}$$

$$a^{n/m} = \sqrt[m]{a^n}$$

$$(a^m)^n = a^{mn}$$

Roots

$$\sqrt{(a \times b)} = \sqrt{a} \times \sqrt{b}$$ $a \in P^+$, $b \in P^+$ ie real and positive

$$\sqrt{(a / b)} = \sqrt{a} / \sqrt{b}$$

$$a / \sqrt{b} = a\sqrt{b} / b$$

$$1 / (a + \sqrt{b}) = (a - \sqrt{b}) / (a^2 - b)$$

Restrictions

$$(ab)^c = a^c b^c$$ $a \in P^+$, $b \in P^+$, $c \in P$

otherwise we get $1^{1/2} = ({}^-1 \times {}^-1)^{1/2} \neq ({}^-1)^{1/2} . ({}^-1)^{1/2} = {}^-1$

Continued Fractions

$(a + \sqrt{b})$ is termed a quadratic irrational if a and b are fractions and b not a perfect square

Continued fraction representation follow the pattern A,B,C…C,B,2A,B,C,…C,B,2A,B,C etc.

eg $\sqrt{14} = \langle 3;1,2,1,6,1,2,1,6,1,2,1,6,,\ldots \rangle$ *(TI-83 accurate to 20 terms)*

by way of interest $\pi = \langle 3;7,15,1,2,9,2,1,1,1,2,1,3,1,\ldots \rangle$ *(TI-83 accurate to 13 terms)*

Notes

Mensuration

circumference of circle	=	$2\pi r$	r = *radius of circle*
length arc of a circle	=	$r\theta$	
length chord of a circle	=	$2r \sin \frac{1}{2}\theta$	
circumference of an ellipse	~	$\pi(a + b)$	*first approximation*
	~	$\pi[\, 3(a+b) - \sqrt{[\, (a+3b)(3a+b) \,]}$]	
area circle	=	πr^2	
area sector	=	$\frac{1}{2} r^2\theta$	
area sector	=	$\frac{1}{2} \int r^2 \, d\theta$	
area of segment	=	$\frac{1}{2} r^2 (\theta - \sin \theta)$	θ *measured in radians*
area ellipse	=	πab	
surface area slice of a sphere	=	$2\pi rh$	*where* h *is spacing*
surface area sphere	=	$4\pi r^2$	*setting* h = 2r
surface area spherical cap	=	$2\pi rk$	*setting* h = k
by setting r	=	$(a^2 + k^2)/_{2k}$	a = *radius base cap*
surface area spherical cap	=	$\pi(a^2 + k^2)$	
total surface area cone	=	πr (r + *slant height*)	
total surface area of a cylinder	=	$2\pi r (h + r)$	*which is rather neat*
volume sphere	=	$^4/_3 \pi r^3$	r = *radius of sphere*
volume sphere cap	=	$^1/_6 \pi h(3a^2 + h^2)$	a = *radius of base*
	=	$^1/_3 \pi k^2(3r - k)$	k = *height of cap*
volume ellipsoid	=	$^4/_3 \pi abc$	
volume cylinder	=	$\pi r^2 h$	
volume pyramid	=	$^1/_3$ (*area base*) × *height*	
volume cone	=	$^1/_3 \pi r^2 h$	
volume rectangular base frustrum	=	$^1/_3 \{ [\, A + B + \sqrt{(A \times B)}] \times h \,] \}$	
		A *and* B *are areas of top and bottom faces*	
volume square base frustrum	=	$^1/_3 h (a^2 + ab + b^2)$	
volume circular cone frustrum	=	$^1/_3 \pi h (R^2 + Rr + r^2)$	

Volumes of the Euclid's 5 Regular Polyhedrons side l

Shape	Volume		Value	Surface Area		Value
tetrahedron	$\sqrt{2}/12$	≈	0.118	$\sqrt{3}$	≈	1.732
cube (hexahedron)	1			6		
octahedron	$\sqrt{2}/3$	≈	0.471	$2\sqrt{3}$	≈	3.464
dodecahedron	$\frac{1}{4}(15+7\sqrt{5})$	≈	7.663	$\sqrt{(225+90\sqrt{5})}$	≈	20.646
icosahedron	$5(3+\sqrt{5})/12$	≈	2.182	$5\sqrt{3}$	≈	8.660

Triangle Trigonometry Rules

where A + B + C = 180°

cosine *rule*: $a^2 = b^2 + c^2 - 2bc \cos A$

sine *rule*: $a / \sin A = b / \sin B = c / \sin C = 2R$ *(circumcircle)*

tangent *rule*: $(a-b)/(a+b) = \tan\frac{1}{2}(A-B) / \tan\frac{1}{2}(A+B)$ $\qquad \tan\frac{1}{2}(A+B) = \cot\frac{1}{2}C$

$\sin\frac{1}{2}A = \sqrt{\{(s-b)(s-c)/bc\}}$ $\qquad$ bc *any two sides*

$\cos\frac{1}{2}A = \sqrt{\{s(s-a)/bc\}}$

$\tan\frac{1}{2}A = \sqrt{\{(s-b)(s-c)/s(s-a)\}}$

$4 \cos\frac{1}{2}A \cos\frac{1}{2}B \cos\frac{1}{2}C = \sin A + \sin B + \sin C$

$4 \sin\frac{1}{2}A \sin\frac{1}{2}B \sin\frac{1}{2}C = \cos A + \cos B + \cos C - 1$

$\tan A \tan B \tan C = \tan A + \tan B + \tan C$

$\cot\frac{1}{2}A \cot\frac{1}{2}B \cot\frac{1}{2}C = \cot\frac{1}{2}A + \cot\frac{1}{2}B + \cot\frac{1}{2}C$

$1 = \tan\frac{1}{2}A \tan\frac{1}{2}B + \tan\frac{1}{2}B \tan\frac{1}{2}C + \tan\frac{1}{2}C \tan\frac{1}{2}A$

Means

start with the *power mean* = $\sqrt[p]{(\frac{1}{n} \sum_{i=1}^{n} a_i^p)}$ $\qquad$ *aka generalised mean*

set p =−1 we get *harmonic mean* = $(\frac{1}{n} \sum_{i=1}^{n} (a_i)^{-1})^{-1}$

set p =0 & we get *geometric mean* = $\sqrt[n]{(\prod_{i=1}^{n} a_i)}$ $\qquad$ *(this is correct)*

set p =1 & we get *arithmetic mean* = $\frac{1}{n} \sum_{i=1}^{n} a_i$

set p =2 & we get *quadratic mean* = $\sqrt[2]{(\frac{1}{n} \sum_{i=1}^{n} a_i^2)}$ $\qquad$ *aka root mean square*

heronian mean = $\frac{1}{3}[a + \sqrt{(ab)} + b]$ $\qquad = \frac{2}{3}$ a.m. + $\frac{1}{3}$ g.m.

arithmetic-geometric and geometric-harmonic means are calculated by an iterative process

Partial Fractions

Where $f(x)$ *is a lesser degree than the denominator.*

Type 1 $\quad f(x)/(x+a).(x-b).(x+c) = A/(x+a) + B/(x-b) + C/(x+c)$

example $\quad 4x^2 + 2x - 14/x^3+3x^2-x-3 = 3/(x+1) - 1/(x-1) + 2/(x+3)$

Type 2 $\quad f(x)/(x+a)^3 = A/(x+a) + B/(x+a)^2 + C/(x+a)^3$

A quick way to do this is to set $x + a = z$ *and then rearrange* $f(x)$

example $\quad 4x+1/(x+1)^3 = 4/(x+1)^2 - 3/(x+1)^3$

Type 3 $\quad f(x)/(ax^2+bx+c) \cdot (cx+d) = Ax + B/(ax^2 bx + c) + C/(cx+d)$

where the expression $ax^2 + bx + c$ *does not factorise.*

example $\quad -x-3/(x^2+1) \cdot (x+1) = x - 1/(x^2 + 1) - 2/(x+1)$

Type 4 $\quad f(x)/(ax^2 + b)^2 \cdot (cx + d) = Ax + B/(ax^2+b)^2 + Cx + D/(ax+b) + E/(cx+d)$

example $\quad 3x - 1/(2x^2 - 1)^2 \cdot (x + 1) = 8x - 5/(2x^2 - 1)^2 + 8(x - 1)/(2x^2 - 1) - 4/(x + 1)$

If $f(x)$ *is of the same degree as* $g(x)$ *then carry out a straight division first.*

Type 5 $\quad f(x)/g(x) = 1 + A / g(x)$ *then proceed as above*

Examples are $\quad x/x+1 = 1 - 1 / x + 1$

$x/x+a = 1 - a / x + a$

Type 6 This principle can be extended to expressions such as $Ax + B + C/f(x) + D/g(x)$

Remainder Theorem

$f(x) \equiv g(x) \times \text{divisor} + \text{remainder}$ *where the divisor is a linear factor*

so $f(x) \equiv g(x) \times (x - a) + R$ *nb the identy sign* $\equiv$

Put $x = a$ *and we get* $f(a) = R$

If a polynomial $f(x)$ *is divided by* $(x - a)$ *then the remainder is* $= f(a)$

If a polynomial $f(x)$ *is divided by* $(bx - a)$ *then the remainder is* $= f(a/b)$

Factor Theorem

If $f(x)$ *is a polynomial and* $f(a) = 0$ *then* $(x - a)$ *is a factor*

If $f(x)$ *is a polynomial and* $f(a/b) = 0$ *then* $(bx - a)$ *is a factor*

Quadratics General Solution with roots α and β

$y = ax^2 + bx + c$

as $x^2 + bx = (x + \frac{1}{2}b)^2 - (\frac{1}{2}b)^2$ *it is a short step to show*

$x = {}^{-b \pm \sqrt{(b^2 - 4ac)}}/_{2a}$

$\Delta = b^2 - 4ac$ *termed the discriminant*

$\alpha + \beta = {}^{-b}/_{a}$

$\alpha\beta = {}^{c}/_{a}$

$\frac{1}{2}(\alpha + \beta) = {}^{-b}/_{2a}$

so line of symmetery is $x = {}^{-b}/_{2a}$ *which is the midpoint of the two roots*

f (x + a) *is a translation of* $^{-}$a *in the x-direction*

f (x) + a *is a translation of* $^{+}$a *in the y-direction*

f (ax) *is a stretch of* $^{1}/_{a}$ *in the x-direction (divide x-coord. by* a*)*

a f (x) *is a stretch of* a *in the y-direction (multiply y-coord. by* a*)*

Complex Solutions of the Quadratic

for $ax^2 + bx + c$ *the roots are* p + iq *and* p − iq *where*

$p = {}^{-b}/_{2a}$

$q = {}^{\sqrt{(4ac - b^2)}} / {}_{2a}$

These solutions hold for ***<u>all</u>*** *quadratic equation with real coefficients.*

Hence if a + bi *is a root of* f(x) *then* a − bi *is also a root and both may be real or complex.*

Particular Solution of a Related Quadratic

if $(x + a)^{\frac{1}{2}} + (x - a)^{\frac{1}{2}} - (x - b)^{\frac{1}{2}} = 0$ *then*

$x = {}^{b \pm 2\sqrt{(b^2 + 3a^2)}}/_{3}$ *the challenge is finding integer sol^{n}.*

Notes on the Complex Conjugate

The conjugate offering a second solution to a quadratic is a specific example of the general.

i *is qualitatively indistinct from its additive and multiplicative inverse* $^{-}$i [eg $i^2 = (^{-}i)^2$]

So for many natural settings if a complex number provides a solution so will its conjugate.

Logarithms

If $a = b^n$

then we define for any given base b $\log_b a = n$

and we define the antilog such that $b^n = \text{antilog}_b\, n$

and so as long as we are in base b $b^{\log a} = a$

and usefully $\log_b b = 1$ (*also* $\ln e = 1$)

1) Product Rule $\log_b xy = \log_b x + \log_b y$

2) Quotient Rule $\log_b({}^x/_y) = \log_b x - \log_b y$

3) Power Rule $\log_b a^k = k \log_b a$

Setting $k = {}^-1$ *gives* $\log_b({}^1/_a) = {}^-\log_b a$

so now we can find the logs of fractions but not negatives in real domain

as before let $b^n = a$

take logs both sides to a different base c $\log_c b^n = \log_c a$

as $n = \log_b a$ $\log_b a \log_c b = \log_c a$

5) Log Product Rule *by rearranging* $\log_c b \log_b a = \log_c a$

and this pattern can be extended to any number of products

6) Base Change Rule *by rearranging again* $\log_b a = \log_c a / \log_c b$

so now we can find the log to any base by setting $c = 10$ *or* e.

7) Power Base Rule *which follows on* $\log_{b^a} c = {}^1/_a \log_b c$

8) Power Base/Inverse Rule *Setting* $a = {}^-1$ *gives* $\log_{1/b} {}^1/_c = \log_b c$

9) Proportionality Rule $\log_b a \quad k \log_c a$

if we set $c = a$ *and remember that* $\log_a a = 1$

10) Reciprocal Rule $\log_b a = (\log_a b)^{-1}$

11) Combination Rule *from Rule 3) and 1)* $\log_b a + k = \log_b (a \times b^k)$

12) Exponential Rule

and recollect as long as we are in base b $a = b^{\log a}$

so setting b = e *gives* $a = e^{\ln a}$

giving the key relationship $a^x = e^{x \ln a}$ †

Notes

There are infinitely many logarithms of a positive number but only one of them is real (Euler)

† *This is the theorem which we will later use to raise complex numbers to complex powers.*

Coordinate Geometry - Linear Equations

$$y = mx + c$$

$$ax + by + c = 0$$

$$m = (y_2 - y_1) / (x_2 - x_1)$$

$$y - y_1 = m(x - x_1)$$

$$(y - y_1) / (y_2 - y_1) = (x - x_1) / (x_2 - x_1)$$

if $m_1 m_2 = {}^{-}1$ *then lines are perpendicular*

midpoint line (x_1, y_1) *to* (x_2, y_2) *is* $\{\tfrac{1}{2}(x_1 + x_2), \tfrac{1}{2}(y_1 + y_2)\}$

distance between (x_1, y_1) *and* (x_2, y_2) *is*

$$\sqrt{\{(x_2 - x_1)^2 + (y_2 - y_1)^2\}}$$

equation of perpendicular bisector of (x_1, y_1) *and* (x_2, y_2) *is*

$$(x_1 - x_2)x + (y_1 - y_2)y = \tfrac{1}{2}\{(x_1^2 - x_2^2) + (y_1^2 - y_2^2)\}$$

Equation circle centre (h,k) *and radius* r *is*

$$(x - h)^2 + (y - k)^2 = r^2$$

Perpendicular distance from (h,k) *to* $ax + by + c = 0$

$$= |ah + bk + c| / \sqrt{(a^2 + b^2)}$$

Acute angle ϕ *between lines with gradients* m_1 *and* m_2

$$= \tan^{-1} (m_1 - m_2) / (1 + m_1 m_2)$$

If $m_1 m_2 = {}^{-}1$ *then the above gives* $\tan^{-1} = \infty$ *and hence* $\phi = 90°$ *ie lines perpendicular*

Coordinate Geometry - Circle

$(x - h)^2 + (y - k)^2 = r^2$ *for circle centre* (h,k) *and radius* r

Notes

as $\dfrac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi} = \tan(\theta - \phi)$ *then you might deduce that*

$\tan^{-1}\left(\dfrac{\theta - \phi}{1 + \theta\phi}\right) = \tan^{-1}\theta - \tan^{-1}\phi$ *which looks logically correct.*

but as tan *isn't a one-to-one function the RHS can be out by* $\pm\ \pi$

Try putting $m_1 = 1$ *and* $m_2 = {}^{-}2$ *and you'll get* ${}^{-}72°$ *(correct) and* $108°$ *(sort of incorrect)*

Area of a Triangle

Sides are a b *and* c, *angles are* A B *and* C.

Further s *is semiperimeter* R *is radius circumcircle and* r *is the radius of the inscribed circle.*

A_Δ	=	$\frac{1}{2}\, a\, b \sin C$	†
A_Δ	=	$\frac{1}{2} \{ a^2 \, {}^{\sin B \sin C} / {}_{\sin(B+C)} \}$	†
A_Δ	=	$\sqrt{[\, s\, (s - a)(s - b)(s - c)]}$	*Heron's formula*
A_Δ	=	$\frac{1}{2}$ *abs det.* $\{1, \bar{1}, 1; x_1, x_2, x_3; y_1, y_2, y_3\}$	
	=	$\frac{1}{2} \, \lvert x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \rvert$	
		the excel function is MDETERM	
A_Δ	=	$s \times r$ *which is pretty neat !*	‡
A_Δ	=	$\frac{1}{4}\, {}^{abc}/{}_{R}$	‡

Area of Quadrilaterals (using term "s" for semi-perimeters)

$$\text{area} = \sqrt{[(s - a)(s - b)(s - c)(s - d) - abcd \times \cos^2 \tfrac{1}{2}(A + C)]} \quad ⊢$$

This is Bretschneider's formula for any quadrilateral. For a cyclic quadrilateral this reduces to

$$\text{area} = \sqrt{[(s - a)(s - b)(s - c)(s - d)]}$$

This is Brahmagupta's formula for any cyclic quadrilateral. For a triangle this reduces to

$$\text{area} = \sqrt{[s(s - a)(s - b)(s - c)]}$$

see above

Notes

† *Equating these two gives us the (obvious) relationship* $\sin A = \sin (B + C)$

‡ *Equating these two gives us the relationship* $R\, r = \frac{1}{2}\, {}^{abc}/{}_{(a+b+c)}$

⊢ A *and* C *are opposite angles.*

Bretschneider's formula is due to Coolidge who derived it from a related relationship

Modelling using Sin and Cos Functions

As our variable is usually t let's use H(t) *for the function rather than* y =

strictly $H(t) = A \sin(\omega t + \phi) + k$

where A = amplitude ½ (max value − min value)

and ω = angular frequency $^{360°}/_{T}$

T *is the* *period* *, the time for one complete cycle*

and ϕ = phase *positive values* ϕ *shift the curve back*

negative values of ϕ *shift the curve forward*

At IB Studies level all curves encountered can be modelled by positive/negative sine / cosine .

sine *curves start at 0,* cosine *curves start at max. Watch out for negatives (inversion of curve)*

and k = axis (centre line) ½ (max value + min value)

Procedure

Look at the data given.

Find the maximum value , the minimum value and the period.

Calculate the amplitude **A***, the frequency* ω *and the axis value* **k***.*

By inspection identify if function is $^{+}$sin $^{-}$sin $^{+}$cos *or* $^{-}$cos

Write down the equation and enter it into your calculator.

Use trace function to find any particular value of **H(t)** *or* **t** *{ie on your calculator* **y** *or* **x***}.*

You need to set the WINDOWS y values to roughly to ± 2A *and x values roughly to* ± 2T

Alternatively use the Table *option on your calculator*

Notes

cosine *is* sine *shifted back* 90° *so* cos x = sin (x + 90)

In higher maths we often work in radians where $2\pi^c = 360°$ *(* ^{c} *means radians, often omitted)*

Linear Interpolations

For function $f(x)$ *find two values that sandwich a root* α

$$a = f(a) \qquad {}^{-}\text{ve value}$$

$$b = f(b) \qquad {}^{+}\text{ve value}$$

then first approximation α *is* a_0 *where*

$$a_0 = \frac{f(b)a - f(a)b}{f(b) - f(a)} \qquad \text{(actually adding)}$$

Trapezium Rule

$$\int_a^b y\,dx = \tfrac{1}{2} h\,[y_0 + 2\,(y_1 + y_2 \ldots + y_{n-1}) + y_n]$$

$$\text{where } h = \frac{b-a}{n}$$

$$\text{and } y_i = f(a + ih)$$

Newton-Raphson Iteration

$$f(x) = 0 : x_{n+1} = x_n - f(x_n) / f'(x_n)$$

Approximations where x is small

$$\sin x \approx x$$

$$\cos x \approx 1 - \tfrac{1}{2} x^2$$

$$\tan x \approx x$$

so for small x *we have* $\sin x \approx \tan x$

$$(1 + x)^{1/2} \approx 1 + \tfrac{1}{2} x - \tfrac{1}{8} x^2$$

for $x \to dx$ *2 terms suffice*

$$\ln(1 + x) \approx 1 - \tfrac{1}{2} x^2$$

$$e^x \approx 1 + x + \tfrac{1}{2} x^2$$

Numerical Solution of Differential Equations

$$\left(\frac{dy}{dx}\right)_0 \approx y_1 - y_0 / h$$

$$\left(\frac{dy}{dx}\right)_0 \approx y_1 - y_{-1} / 2h$$

$$\left(\frac{d^2y}{dx^2}\right)_0 \approx y_1 - 2y_0 + y_{-1} / 2h$$

Sequences and Sums - find the next sequence and sum

5) Square Numbers Pattern

$$3^2 + 4^2 = 5^2$$
$$10^2 + 11^2 + 12^2 = 13^2 + 14^2$$
$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$$
$$36^2 + 37^2 + 38^2 + 39^2 + 40^2 = 41^2 + 42^2 + 43^2 + 44^2$$

What comes next?

6) Triangular and Square Numbers

$$0 + 1 = 1^2 \qquad S_1 = 1$$
$$1 + 3 = 2^2 \qquad S_2 = 4$$
$$3 + 6 = 3^2 \qquad S_3 = 9$$
$$6 + 10 = 4^2 \qquad S_4 = 16$$

What comes next?

7) Triangular Numbers Pattern

$$1 + 3 + 6 = 10$$
$$15 + 21 + 28 + 36 = 45 + 55$$
$$66 + 78 + 91 + 105 + 120 = 136 + 153 + 171$$

What comes next?

8) Cubes and Triangular Numbers

$$1^3 = 1^2 \qquad (T_1)^2$$
$$1^3 + 2^3 = 3^2 \qquad (T_2)^2$$
$$1^3 + 2^3 + 3^3 = 6^2 \qquad (T_3)^2$$
$$1^3 + 2^3 + 3^3 + 4^3 = 10^2 \qquad (T_4)^2$$

What comes next?

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Domain and Range					Domain is R		Codomain is R	
Function		**inj**	**sur**	**T**	**x∈**		**Image y∈**	
sin x	odd	$\times_1$	$\times_5$	2π	$(^-\infty, ^+\infty)$		$[^-1, ^+1]$	
$\sin^{-1} x$	odd	$\checkmark_2$	$\times_3$	$\times$	$[^-1, ^+1]$		$[^-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi]$	
cosec x	odd	$\times_1$	$\times_5$	2π	$(^-\infty, ^+\infty)$	#1	$(^-\infty, ^-1] \cup [1, ^+\infty)$	
$\text{cosec}^{-1} x$	odd	$\checkmark_2$	$\times_3\times_4$	$\times$	$(^-\infty, ^-1] \cup [1, ^+\infty)$		$(^-\pi, 0) \cup (0, \pi)$	
cos x	even	$\times_1\times_2$	$\times_2\times_5$	2π	$(^-\infty, ^+\infty)$		$[^-1, ^-1]$	
$\cos^{-1} x$	$\times$	$\checkmark_2$	$\times_3$	$\times$	$[^-1, ^-1]$		$[0, \pi]$	
sec x	even	$\times_1\times_2$	$\times_2\times_5$	2π	$(^-\infty, ^+\infty)$	#2	$(^-\infty, ^-1] \cup [1, ^+\infty)$	
$\sec^{-1} x$	$\times$	$\checkmark_2$	$\times_3$	$\times$	$(^-\infty, ^-1] \cup [1, ^+\infty)$		$[0, \tfrac{1}{2}\pi) \cup (\tfrac{1}{2}\pi, \pi]$	
tan x	odd	$\times_1$	$\checkmark_1$	2π	$(^-\infty, ^+\infty)$	#2	$(^-\infty, ^+\infty)$	
cot x	odd	$\times_1$	$\checkmark_1$	2π	$(^-\infty, ^+\infty)$	#1	$(^-\infty, ^+\infty)$	
$\tan^{-1} x$	odd	$\checkmark_2$	$\times_3$	$\times$	$(^-\infty, ^+\infty)$		$(^-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi)$	
$\cot^{-1} x$	?	$\checkmark_2$	$\times_3$	$\times$	$(^-\infty, 0) \cup (0, ^+\infty)$	#3	$(0, \pi)$	#4
sinh x	odd	$\checkmark_3$	$\checkmark_1$	$\times$	$(^-\infty, ^+\infty)$		$(^-\infty, ^+\infty)$	
$\sinh^{-1} x$	odd	$\checkmark_3$	$\checkmark_1$	$\times$	$(^-\infty, ^+\infty)$		$(^-\infty, ^+\infty)$	
cosech x	odd	$\checkmark_3$	$\times_4$	$\times$	$(^-\infty, 0) \cup (0, ^+\infty)$	#3	$(^-\infty, 0) \cup (0, ^+\infty)$	
$\text{cosech}^{-1} x$	odd	$\checkmark_3$	$\times_4$	$\times$	$(^-\infty, 0) \cup (0, ^+\infty)$	#3	$(^-\infty, 0) \cup (0, ^+\infty)$	
cosh x	even	$\times_2$	$\times_2$	$\times$	$(^-\infty, ^+\infty)$		$[1, ^+\infty)$	
$\cosh^{-1} x$	$\times$	$\checkmark_4$	$\times_3$	$\times$	$[1, ^+\infty)$		$[0, ^+\infty)$	
sech x	even	$\times_2$	$\times_2$	$\times$	$(^-\infty, ^+\infty)$		$(0, ^+1]$	
$\text{sech}^{-1} x$	$\times$	$\checkmark_4$	$\times_3$	$\times$	$(0, ^+1]$		$[0, ^+\infty)$	
tanh x	odd	$\checkmark_3$	$\times_6$	$\times$	$(^-\infty, ^+\infty)$		$(^-1, ^+1)$	
$\tanh^{-1} x$	odd	$\checkmark_3$	$\checkmark_1$	$\times$	$(^-1, ^+1)$		$(^-\infty, ^+\infty)$	
coth x	odd	$\checkmark_3$	$\times$	$\times$	$(^-\infty, 0) \cup (0, ^+\infty)$	#3	$(^-\infty, ^-1) \cup (1, ^+\infty)$	
$\coth^{-1} x$	odd	$\checkmark_3$	$\times_4$	$\times$	$(^-\infty, ^-1) \cup (1, ^+\infty)$		$(^-\infty, 0) \cup (0, ^+\infty)$	

#1 $x \neq n\pi$ #2 $x \neq \tfrac{1}{2}\pi + n\pi$ #3 $(^-\infty, 0) \cup (0, ^+\infty) \equiv x \in R\ x \neq 0$

#4 *This definition gives a continuous function. TI-83 will plot function in 3rd quadrant not 2nd*

rg

Sequences and Sums - find the next sequence and sum

13) Cubes and Differences Triangular Numbers

$2^3 = 3^2 - 1^2 \quad (T_2 - T_1)$

$3^3 = 6^2 - 3^2 \quad (T_3 - T_2)$

$4^3 = 10^2 - 6^2 \quad (T_4 - T_3)$

$5^3 = 15^2 - 10^2 \quad (T_5 - T_4)$

What comes next?

14) Sum Cubes and Product Squares

$1^3 = \frac{1}{4} \times 1^2 \times 2^2$

$1^3 + 2^3 = \frac{1}{4} \times 2^2 \times 3^2$

$1^3 + 2^3 + 3^3 = \frac{1}{4} \times 3^2 \times 4^2$

$1^3 + 2^3 + 3^3 + 4^3 = \frac{1}{4} \times 4^2 \times 5^2$

What comes next?

15) Centred Triangular Numbers

$3 \times 1 + 1 = 4 \quad T_{c2}$

$3 \times 3 + 1 = 10 \quad T_{c3}$

$3 \times 6 + 1 = 19 \quad T_{c4}$

$3 \times 10 + 1 = 31 \quad T_{c5}$

What comes next?

16) Perfect Numbers (sum factors = 2n)

$1 + 2 + 3 = 6$

$1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$

Double last number sequence and add 1. If prime then sequence sum is perfect number.

7x2+1 = 15. Not prime. 15 x 2 + 1 = 31. Prime.

$1 + 2 + 3 + 4 + 5 + \ldots + 31 = 496$

31x2+1 = 63. Not prime. 63 x 2 + 1 = 127. Prime.

$1 + 2 + 3 + 4 + 5 + \ldots + 127 = 8128$

What comes next? (This is tricky)

Conics	Ellipse	Parabola	Hyperbola	Rectangular Hyperbola
Standard Form	$x^2/a^2 + y^2/b^2 = 1$	$y^2 = 4ax$	$x^2/a^2 - y^2/b^2 = 1$	$xy = c^2$
Parametric Form	$(a\cos\theta, b\sin\theta)$	$(at^2, 2at)$	$(a\sec\theta, b\tan\theta)$ $(\pm a\cosh\theta, b\sinh\theta)$	$(ct, c/t)$
Eccentricity	$e < 1$ $b^2 = a^2(1-e^2)$	$e = 1$	$e > 1$ $b^2 = a^2(e^2-1)$	$e = \sqrt{2}$
Foci	$(\pm ae, 0)$	$(a, 0)$	$(\pm ae, 0)$	$(\pm\sqrt{2}c, \pm\sqrt{2}c)$
Directrices	$x = \pm a/e$	$x = -a$	$x = \pm a/e$	$x + y = \pm\sqrt{2}c$
Asymptotes	none	none	$x/a = \pm y/b$	$x = 0, y = 0$

Curvature

$$\rho = \kappa = ds/d\psi$$

Arc Length

$$s = \int_{xA}^{xB} \sqrt{[1+(dy/dx)^2]}\, dx$$ *cartesian coord.*

$$\text{OR } s = \int_{yA}^{yB} \sqrt{[1+(dx/dy)^2]}\, dy$$ *cartesian coord.*

$$\text{OR } s = \int_{tA}^{tB} \sqrt{[(dx/dt)^2+(dy/dt)^2]}\, dt$$ *parametric form*

$$s = \int \sqrt{\{r^2+(dr/d\theta)^2\}}\, d\theta$$ *polar coord.*

Surface Area of Revolution

$$s_x = \int_{xA}^{xB} 2\pi y \sqrt{[1+(dy/dx)^2]}\, dx$$ *cartesian coord.*

$$s_y = \int_{yA}^{yB} 2\pi x \sqrt{[1+(dx/dy)^2]}\, dx$$ *cartesian coord.*

$$= \int_{tA}^{tB} 2\pi y \sqrt{[(dx/dt)^2+(dy/dt)^2]}\, dt$$ *parametric form*

Notes

Theory of Equations

If we have $(x + a_1)(x + a_2)(x + a_3)(x + a_4) \dots (x + a_n)$ *the expansion is*

$x^n + (a_1 + a_2 + a_3 + a_4 \dots a_n)\, x^{n-1}$

$+ (a_1 a_2 + a_1 a_3 + \dots a_1 a_n + a_2 a_3 + a_2 a_4 \dots a_2 a_n + \dots a_{n-1} a_n) x^{n-2}$

$+ (a_1 a_2 a_3 + a_1 a_2 a_4 + \dots a_1 a_2 a_n + a_1 a_3 a_4 + a_1 a_3 a_5 + \dots a_1 a_3 a_n + \dots a_{n-1} a_{n-1} a_n) x^{n-3}$

$+ (a_1 a_2 a_3 a_4 + \dots a_{n-2} a_{n-1} a_n)$

so if the original coefficients are real then both sum and product all roots must be real.

so (I have read but don't quite see why) there must be at least one pair of roots such that

$$(a + ib) + (c + id) = p + 0i$$

$$(a + ib) \cdot (c + id) = q + 0i$$

it is then easy to show that $a = c$ and $b = {}^{-}d$

so all the roots can be put into conjugate pairs.

This is a sufficient condition but not a complete proof that it is a necessary condition.

If there is an odd number of roots the remaining root must be real.

Graphically algebraic functions of odd order must cross the x-axis somewhere.

If I multiply out $(x + a + ib)(x + a - ib)(x + c + id)(x + c - id)$ *I get all real coeff.*

$x^4+[2a+2c]x^3+[a^2+b^2+c^2+d^2+4ac]x^2+[(2c)(a^2+b^2)+(2a)(c^2+d^2)]x+[a^2c^2+a^2d^2+b^2c^2+b^2d^2]$

so if I then hypothesise solutions $(1 + 2i)\ (1 - 2i)\ (3 + 4i)\ (3 - 4i)$ *to a quartic equation*

substituting these values for x into equ. $x^4 + 8x^3 + 42x^2 + 80x + 125$ *gives 0 each time.*

So now I think I understand why solutions to equations must come in conjugate pairs.

Notes

i *is qualitatively indistinct from its additive and multiplicative inverse* ${}^{-}i$ [eg $i^2 = ({}^{-}i)^2$]

So for many natural settings if a complex number provides a solution so will its conjugate.

This also explains the restriction when splitting surds. Numbers must be real and positive because the mathematics cannot distinguish between ${}^{-}i$ *and* ${}^{+}i$

Orders of Magnitude

septillionth	yocto-	y	10^{-24}	septillion	yotta-	Y	10^{24}
sextillionth	zepto-	z	10^{-21}	sextillion	zetta-	Z	10^{21}
quintillionth	atto-	a	10^{-18}	quintillion	exa-	E	10^{18}
quadrillionth	femto-	f	10^{-15}	quadrillion	peta-	P	10^{15}
trillionth	pico-	p	10^{-12}	trillion	tera-	T	10^{12}
billionth	nano-	n	10^{-9}	billion	giga-	G	10^{9}
millionth	micro-	μ	10^{-6}	million	mega-	M	10^{6}
thousandth	milli-	m	10^{-3}	thousand	kilo-	k	10^{3}
hundredth	centi-	c	10^{-2}	hundred	hecto-	h	10^{2}
tenth	deci-	d	10^{-1}	ten	deca-	da	10^{1}
one	-	-	10^{0}	one	-	-	10^{0}

Mathematical Constants - 30 decimals (last place not rounded)

pi	π	=	*3.14159 26535 89793 23846 26433 83279...*
exponential	e	=	*2.71828 18284 59045 23536 02874 71352...*
Pythagoras's	$\sqrt{2}$	=	*1.41421 35623 73095 04880 16887 24209...*
	$\sqrt{3}$	=	*1.73205 08075 68877 29352 74463 41505...*
	$\log 2$	=	*0.69314 71805 59945 30941 72321 21458...*
golden ratio	ϕ	=	*1.61803 39887 49894 84820 45868 34365...*
Euler-Mascheroni	γ	=	*0.57721 56649 01532 86060 65120 90082...*
Feigenbaum's	δ	=	*4.66920 16091 02990 67185 32038 20466...*
	$\xi(2)$	=	*1.64493 40668 48226 43647 24151 66646...*
Apery's	$\xi(3)$	=	*1.20205 69031 59594 28539 97381 61511...*
	$\xi(4)$	=	*1.08232 32337 11138 19151 60036 96541...*
Euler's	$\xi(5)$	=	*1.03692 77551 43369 92633 13654 86457...*
	$\xi(6)$	=	*1.01734 30619 84449 13971 45179 29790...*
	e^{π}	=	*23.14069 26327 79269 00572 90863 67948...*

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Prime Numbers (in columns of 25)

2	101	233	383	547	701	877	1049	1229	1429	1597	1783
3	103	239	389	557	709	881	1051	1231	1433	1601	1787
5	107	241	397	563	719	883	1061	1237	1439	1607	1789
7	109	251	401	569	727	887	1063	1249	1447	1609	1801
11	113	257	409	571	733	907	1069	1259	1451	1613	1811
13	127	263	419	577	739	911	1087	1277	1453	1619	1823
17	131	269	421	587	743	919	1091	1279	1459	1621	1831
19	137	271	431	593	751	929	1093	1283	1471	1627	1847
23	139	277	433	599	757	937	1097	1289	1481	1637	1861
29	149	281	439	601	761	941	1103	1291	1483	1657	1867
31	151	283	443	607	769	947	1109	1297	1487	1663	1871
37	157	293	449	613	773	953	1117	1301	1489	1667	1873
41	163	307	457	617	787	967	1123	1303	1493	1669	1877
43	167	311	461	619	797	971	1129	1307	1499	1693	1879
47	173	313	463	631	809	977	1151	1319	1511	1697	1889
53	179	317	467	641	811	983	1153	1321	1523	1699	1901
59	181	331	479	643	821	991	1163	1327	1531	1709	1907
61	191	337	487	647	823	991	1171	1361	1543	1721	1913
67	193	347	491	653	827	1009	1181	1367	1549	1723	1831
71	197	349	499	659	829	1013	1187	1373	1553	1733	1933
73	199	353	503	661	839	1019	1193	1381	1559	1741	1949
79	211	359	509	673	853	1021	1201	1399	1567	1747	1951
83	223	367	521	677	857	1031	1213	1409	1571	1753	1973
89	227	373	523	683	859	1033	1217	1423	1579	1759	1979
97	229	379	541	691	863	1039	1223	1427	1583	1777	1999

Notes

Prime Number Theorem states that the number of primes up to n, $\pi_n \sim {}^{n}/_{\ln(n)}$

Alternatively the nth *prime number* $p_n \sim n \ln(n)$. *So* $p_{300} \sim 300 \ln 300 = 1711$ *(cf* 1999*)*

If $\mathrm{li} = \int {}^{dt}/_{\ln t}$ *then* $\mathrm{Li}(x) = {}_2\!\int^x {}^{dt}/_{\ln t} = \mathrm{li}(x) - \mathrm{li}(2)$ *is a better approximation to* $\pi(x)$

Goodhand's conjecture states the percent proportion of primes approximately equals the percent that ${}^{n}/_{\ln(n)}$ *underestimates* $p(n)$. *Hence* $\pi(n)$ *better* $\approx \tfrac{1}{2}(1 - \sqrt{\{1 - {}^{4}/_{\ln(n)}\}})$

Counting

No.	***Greek***	***Latin***
1	*mono*	*uni*
2	*duo*	*bi*
3	*tri*	*tri*
4	*tetra*	*quad*
5	*penta*	*quin*
6	*hexa*	*sex*
7	*hepta*	*sept*
8	*octo*	*oct*
9	*nona*	*non*
10	*deca*	*dec*

These booklets are written and produced by Robert Goodhand

Although the formulae and expressions given have been individually derived and checked errors do creep in. The booklets are also continuously updated.

If you would like the latest issue, just email me at robert.goodhand@gmail.com

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