# Arithmo-Geometric Series

## *Derivation*

Suppose I have an arithmo-geometric series of six terms

$S_6 = a + ( a + d ) r + ( a + 2d ) r^2 + ( a + 3d ) r^3 + ( a + 4d ) r^4 + ( a + 5d ) r^5$

$= a + ar + ar^2 + ar^3 + ar^4 + ar^5 + dr + 2dr^2 + 3dr^3 + 4dr^4 + 5dr^5$

$rS_6 = ar + ar^2 + ar^3 + ar^4 + ar^5 + ar^6 + dr^2 + 2dr^3 + 3dr^4 + 4dr^5 + 5dr^6$

$S_6 - rS_6 = ( a - ar^6 ) + dr + dr^2 + dr^3 + dr^4 + dr^5 - 5dr^6$

$= ( a - ar^6 ) + d ( r + r^2 + r^3 + r^4 + r^5 ) - 5dr^6$

$S_6 ( 1 - r ) = a ( 1 - r^6 ) + dr \{ ( 1 - r^5 ) / ( 1 - r ) \} - 5dr^6$

and I can see that in general

$S_n (1 - r) = a ( 1 - r^n ) / ( 1 - r ) + dr ( 1 - r^{n-1} ) / ( 1 - r )^2 - ( n - 1 ) dr^n / ( 1 - r )$

which doesn't look particularly simpler.

However if $r < 1$ then

$S_\infty = a / ( 1 - r ) + dr / ( 1 - r )^2$

which is a bit neater.

## *Example*

Let $S_5 = 3 + 5x + 7x^2 + 9x^3 + 11x^4$

giving a = 3 d = 2 and r = x. So from our formula

Let $S_5 = 3( 1 - x^5) / ( 1 - x ) + 2x ( 1 - x^4 ) / ( 1 - x )^2 + 8x^5 /( 1 - x )$

which seems to move us from bad to worse. However if I enter onto my TI-83 as Y1 and Y2 I can see that the functions are equivalent except Y2 is undefined at x = 1 as expected. To test my infinite sum I try

$S = 3 + {}^5/_2 + {}^7/_4 + {}^9/_8 + {}^{11}/_{16} + {}^{13}/_{32} \ldots$

but my formula gives $3/ \frac{1}{2} + 1 / (\frac{1}{2})^2 = 10$

If I sum the series to 11 terms I get 9.97 which seems a fair indication correct.

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