Bayes Theorem

Suppose we shake one dice. The dice could land One, two ,three , four, five , six. If we shake the dice a large number of times the proportion of say "five" to the total number of throws approaches 1/6.

We therefore say P("five") = 1/6.

The probability of a "five" is one in six.

Suppose we shake two die. We can now chart out the possible outcomes.

	1	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Now suppose we wanted the probability that the total was greater than 8.

	1	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	**9**
4	5	6	7	8	**9**	**10**
5	6	7	8	**9**	**10**	**11**
6	7	8	**9**	**10**	**11**	**12**

P (total > 8) = $^{10}/_{36}$ *(leave in this form)*

Now suppose we wanted the probability that either die showed a "five"

	1	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	**6**	7
2	3	4	5	6	**7**	8
3	4	5	6	7	**8**	9
4	5	6	7	8	**9**	10
5	**6**	**7**	**8**	**9**	**10**	**11**
6	7	8	9	10	**11**	12

Remembering that in probability OR means "either or both"

P (either shows 5) = $^{11}/_{36}$

Now let's define P (A/B) as meaning the probability of A happening given that we know that B has already happened.

So if we let A = "total > 8"

and B = "either shows 5"

First consider how many outcomes are both "total > 8" AND "either shows 5"

	1	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	**9**	10
5	6	7	8	**9**	**10**	**11**
6	7	8	9	10	**11**	12

There are only five throws that meet both.

So P (total > 8 / either shows 5) = $^5/_{11}$

But P (either shows 5 / total > 8) = $^5/_{10}$

Bayes Theorem

Note that

P (total > 8 AND either shows 5) = $^5/_{36}$

but also equals

P (total > 8 / either shows five) ×

P (either shows 5)

$^5/_{11} \times {}^{11}/_{36} = {}^5/_{36}$ as expected.

But we can also turn around

P (either shows 5 AND total > 8) = $^5/_{36}$

but also equals

P (either shows 5 / > 8) × P (total > 8)

$^5/_{10} \times {}^{10}/_{36} = {}^5/_{36}$ again as expected.

But now we equate the two

P (total > 8 / either shows five) ×

P (either shows 5)

= P (either shows 5 / total > 8) ×

P (total > 8)

Now to make this clearer let

P (over 8) = P (A)

P (either 5) = P (B)

so we can rearrange to give

P(A/B) = P(B/A) × P(A) / P(B)

which is **Bayes Theorem**

Practical Outcomes

Suppose we said P(A) = probability covid

and P(B) = positive on a test

We want nearly everyone who has covid to show up on the test so we design P(B/A) = 99/100 so few slip through.

But suppose P(A) = 1/5

so P(B) $\cong$ 25/100

P(A/B) = P(B/A) × P(A) / P(B)

= 95/100 × 1/5 / 99/100 = 19/99 $\cong$ 20%

That is by making sure the test "catches everyone" who does have covid that doesn't mean being positive on the test means you really do have covid. There's a world of difference between testing positive given you really do have covid and you really do have covid given you've tested positive. Setting the bar high means a lot of false positives will occur. This phenomena becomes particularly noticeable when you try and test for something that has a relatively low incidence of occurring in the population.

Bayes theorem will show that even when you get a positive result the probability that you actually have the condition is still very low and further tests are required.

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Appendix Further Proof

This is the proof I ws given when undertaking a postgraduate certificate.

Define P (A/E) as the probability A given E

Now assume the probability of A occurring given that event E has already occurred is some function of probability of A **AND** E occurring say k.

We define P(A **AND** E) as $P(A \cap E)$ termed *intersection*.

Terefore $P(A/E) = k\, P(A \cap E)$

Now $P(A/A) = k\, P(A \cap A) = kP(A) = 1$

So $k = {}^{1}/_{P(A)}$ and we immediately derive

$P(A/E) = {}^{P(A \cap E)} / {}_{P(E)}$

P(A) is termed the prior probability

E is additional information probability P(E)

P(A/E) is the posterior probability.