Calculating Bernoulli Numbers

The recurrence formula is

$\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0$ with seed $B_0 = 1$

The practicality of this is that we set up the pseudo binomial expansion of $(B+1)^n$, eliminate the first term and equate what remains to xero. Substituting previously known B numbers determines the next.

So $(B+1)^2 = B^2 + 2B + 1$

Let $2B + 1 = 0$ giving $\mathbf{B_1 = -1/2}$

$(B+1)^3 = B^3 + 3B^2 + 3B + 1$

Set $3B^2 + 3B + 1 = 0$

So $3B^2 + 3(-1/2) + 1 = 0$

giving $\mathbf{B_2 = 1/6}$

$(B+1)^4 = B^4 + 4B^3 + 6B^2 + 4B + 1$

Set $4B^3 + 6B^2 + 4B + 1 = 0$

So $4B^3 + 6\,(1/6) + 4\,(-1/2) + 1 = 0$

giving $\mathbf{B_3 = 0}$ *perhaps surprisingly.*

$(B+1)^5$

$= B^5 + 5B^4 + 10B^3 + 10B^2 + 5B + 1$

Set $5B^4 + 10B^3 + 10B^2 + 5B + 1 = 0$

So $5B_4 + 10\,(0) + 10\,(1/6)$

$+ 5\,(-1/2) + 1 = 0$

giving $\mathbf{B_4 = -1/30}$

$(B+1)^6 = B^6 + 6B^5 + 15B^4$

$+ 20B^3 + 15B^2 + 6B + 1$

Set $6B^5 + 15B^4 + 20B^3$

$+ 15B^2 + 6B + 1 = 0$

So $6B_5 + 15\,(-1/30) + 20\,(0)$

$+ 15\,(1/6) + 6\,(-1/2) + 1 = 0$

giving $\mathbf{B_5 = 0}$

$(B+1)^7 = B^7 + 7B^6 + 21B^5 + 35B^4 +$

$235B^3 + 21B^2 + 7B + 1$

Set $7B^6 + 21B^5 + 35B^4$

$+35B^3 + 21B^2 + 7B + 1 = 0$

So $7B_6 + 21\,(0) + 35\,(-1/30) + 35\,(0)$

$+ 21\,(1/6) + 7\,(-1/2) + 1 = 0$

giving $\mathbf{B_6 = 1/42}$

$(B+1)^8 = B^8 + 8B^7 + 28B^6 + 56B^5 +$

$70B^4 + 56B^3 + 28B^2 + 8B + 1$

Set $8B^7 + 28B^6 + 56B^5 + 70B^4 +$

$56B^3 + 28B^2 + 8B + 1 = 0$

So $8B_7 + 28\,(1/42) + 56(0) + 70\,(-1/30)$

$+ 56(0) + 28\,(1/6) + 8\,(-1/2) + 1 = 0$

giving $\mathbf{B_7 = 0}$ as expected

$(B+1)^9 = B^9 + 9B^8 + 36B^7 + 84B^6$

$+ 126B^5 + 126B^4 + 84B^3$

$+ 36B^2 + 9B + 1 = 0$

From hereon I'm omitting the odd

terms in the expansion

So $9B_8 + 84\,(^1/_{42}) + 126\,(^{-1}/_{30})$

$+ 36\,(^1/_6) + 9\,(^-½) + 1 = 0$

giving $\mathbf{B_8 = -^1/_{30}}$

B_9 is assumed to be = 0

$(B+1)^{11} = B^{11} + 11B^{10} + 55B^9 + 165B^8$

$+ 330B^7 + 462B^6 + 462B^5 + 330B^4$

$+ 165B^3 + 55B^2 + 11B + 1 = 0$

So $11\,B_{10} + 165\,(^{-1}/_{30}) + 462\,(^1/_{42})$

$+ 330\,(^{-1}/_{30}) + 55\,(^1/_6)$

$+ 11\,(^-½) + 1 = 0$

giving $\mathbf{B_{10} = {}^5/_{66}}$

$(B+1)^{13} = B^{13} + 13B^{12} + 78B^{11} +$

$286B^{10} + 715B^9 + 1287B^8 + 1716B^7$

$+ 1716B^6 + 1287B^5 + 715B^4 + 286B^3$

$+ 78B^2 + 13B + 1 = 0$

So $13B_{12} + 286\,(^5/_{66}) + 1287\,(^{-1}/_{30})$

$+ 1716\,(^1/_{42}) + 715\,(^{-1}/_{30}) + 78\,(^1/_6)$

$+ 13\,(^-½) + 1 = 0$

giving $\mathbf{B_{12} = {}^{-691}/_{2730}}$

and truncating the initial expansion we have

$15B_{14} + 455\,(^{-691}/_{2730}) + 3003\,(^5/_{66})$

$+ 6435(^{-1}/_{30}) + 5005(^1/_{42})$

$+ 1365(^{-1}/_{30}) + 105(^1/_6)$

$+ 15(^-½) + 1 = 0$

giving $\mathbf{B_{14} = {}^7/_6}$

which is perhaps also surprising

Patterns of Bernoulli Numbers

The following patterns are observed

B_n is always rational.

$B_{2n+1} = 0$ for $n \ge 1$.

B_{2n} alternates sign

$B_{4n} < 0$ and $B_{4n+2} > 0$ for $n \ge 1$

The magnitude of B_{2n} grows very quickly

Occurrence of Bernoulli Numbers

In the formula for power series the Bernoulli number is always the last term.

eg $\sum n^2 = {}^1/_3\, n^3 + ½\, n^2 + {}^1/_6\, n$

The formula for the squared power series ends in the 2nd Bernoulli number.

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