Mr G's Little Book on

Beyond Pascal - Multinomials

and a serendipitous exercise in dice throwing

Preface

This booklet is written in two parts.

The first part is a narrative, subsequently published, on a secondary school exercise in dice throwing which led to the discovery of multinomials.

The second part is the formal mathematical exercise to develop Pascal's triangle beyond binomial expansions.

Part 1 Dice Throwing

Year 9 Bottom Set were looking a bit unenthusiastic so I decided to fall back on an exercise I'd developed during my PGCE course – dice throwing.

The exercise went well, and after the usual "blowing for luck" and excessive shaking, the class reluctantly agreed that each number 1 to 6 had the same chance of being thrown. Then we moved on to two dice – throwing them 100 times and plotting a bar chart. Finally we drew up a probability space diagram and counted the occurrences of all totals, comparing them to the experimental values.

Sensing a return to decimal addition, Adam asked *"what about three dice?"*

Now I had taught statistical process control in industry and had always said that the normal distribution would be approximated by throwing many dice but had never really given the matter anymore thought.

"Why not?" I said *"but let's concentrate on the probability space diagram"*. But how to draw one in three dimensions? We compromised and drew six diagrams. Above each one we labelled "Third die = 1" "Third die = 2" etc. Then we just added that value to each square in the probability space. So, for example for "Third die = 3" we got

Third die = 3

	1	2	3	4	5	6
1	5	6	7	8	9	10
2	6	7	8	9	10	11
3	7	8	9	10	11	12
4	8	9	10	11	12	13
5	9	10	11	12	13	14
6	10	11	12	13	14	15

Then we worked through all six probability space diagrams, counting up on a big tally chart on the board how many times "3" occurred, "4" occurred etc., finally drawing our new bar chart running from total "3" to "18". This looked even more like the familiar bell shaped normal distribution and I finished the mini investigation by giving some every day examples of where these occur, such as dartboards and butchers' chopping slabs.

Still hoping to forestall decimal addition, there was a request for four dice but I demurred. Back home though I started to think how I might have tackled it. How quickly could I produce probability space diagrams for many dice? We'd all noticed the distinct patterns that had occurred within each of the six diagrams. Eventually, after some experimenting, I drew up this pattern on a spreadsheet.

The first column is the total of the 3 dice. The next 6 columns are the number of times the third die =1, =2 =3 etc... The final column is the number of times that total occurs.

	=1	*=2*	*=3*	*=4*	*=5*	*=6*	**Total**
3	1						**1**
4	2	1					**3**
5	3	2	1				**6**
6	4	3	2	1			**10**
7	5	4	3	2	1		**15**
8	6	5	4	3	2	1	**21**
9	5	6	5	4	3	2	**25**
10	4	5	6	5	4	3	**27**
11	3	4	5	6	5	4	**27**
12	2	3	4	5	6	5	**25**
13	1	2	3	4	5	6	**21**
14		1	2	3	4	5	**15**
15			1	2	3	4	**10**
16				1	2	3	**6**
17					1	2	**3**
18						1	**1**

What we had realised in class was that the total of say "3" occurs three times in the first probability space diagram *(third die = 1)*, then twice in the next probability space diagram *(third die = 2)* and only once in the third. What a beautiful pattern! I didn't have to draw out all the diagrams at all – just take a column of numbers for frequencies of totals using two dice and stagger them in a diagonal pattern. Add up each row and there's the correct frequency for that total.

The critical test was to come. Could I repeat the trick for four dice? It turned out to be simplicity itself. Here it is.

4	1						**1**
5	3	1					**4**
6	6	3	1				**10**
7	10	6	3	1			**20**
8	15	10	6	3	1		**35**
9	21	15	10	6	3	1	**56**
10	25	21	15	10	6	3	**80**
11	27	25	21	15	10	6	**104**
12	27	27	25	21	15	10	**125**
13	25	27	27	25	21	15	**140**
14	21	25	27	27	25	21	**146**
15	15	21	25	27	27	25	**140**
16	10	15	21	25	27	27	**125**
17	6	10	15	21	25	27	**104**
18	3	6	10	15	21	25	**80**
19	1	3	6	10	15	21	**56**
20		1	3	6	10	15	**35**
21			1	3	6	10	**20**
22				1	3	6	**10**
23					1	3	**4**
24						1	**1**
				Total			**1296**

I was satisfied that my final frequency table was correct – not least that the total of totals came to 1296 which was 6^4, the number of combinations of four dice That was about the end of it, I thought. I now knew I could repeat the pattern for any number of dice. I showed the results to a colleague and she agreed that it did look pretty neat, but I filed it away and thought no more.

Then a few months later a magazine appeared in my in-tray **"Tempus"** *"an occasional magazine produced by the University of West of England School of Mathematics…."* On the back page was this problem.

Enigma 8.

"An ordinary cubical die is tossed five times. What is the probability that the total score is 13?"

This seemed a challenge. First I tried partitioning 13 but that proved intractable. Then I turned to standard textbooks. Here was a surprise. There were plenty of problems throwing two dice solved with the binomial formula but nothing more. I'd always assumed that there were trinomials and higher that would solve such problems in a single line. But here was a problem titled "Enigma", clearly designed to challenge higher degree students so I hypothesised it was unlikely to be solved with a standard formula. Then I remembered my investigation.

Back on the spreadsheet I copied my line and staggered it six times to produce a five dice solution. Straight away I could read off "420" as the number of occurrences of the total 13 in 6^5 (7776) possible outcomes. But it had taken quite a bit of spreadsheet space and solutions were required *"on a postcard"*. So this is what I wrote.

The number of combinations for all totals for n dice can be found by summing up to six terms of the previous sequence. ie

2dice 1 2 3 4 5 6 5 4 3 2 1

3dice 1 3 6 10 15 21 27 27 25 21…

4dice 1 4 10 20 35 56 80 104 125 …

5dice 1 5 15 35 70 126 205 305 420 …

Total '5' '6' '7' '8' '9' '10' '11' '12' '13'

The probability is therefore 420 / 6^5 = 0.05.

which actually fitted onto half a postcard.

At this point I noticed that the sequences initially followed triangular numbers and their higher orders. So for a moment I thought I could simply calculate the answers from a combination formula because I knew there was a direct relationship between the two. However $^{12}C_4$ annoyingly gave 495 not the required 420. Then I realised this was because my terms only summed back six places.

Triangular numbers and their higher orders spiral ever higher but my sequences had to swing back into a symmetrical pattern.

I submitted the requisite postcard and Jim Gowers from U.W.E. contacted me to confirm my solution was correct. He also gave me some further pointers to show how the distribution approximates to the binomial.

Part 2 Beyond Pascal

Hopefully some will have a passing familiarity with the binomial theorem.

We start with

$(1 + x)^2 = 1 + 2x + x^2$

and then move on progressively to

$(1 + x)^3 = 1 + 3x + 3x^2 + x3$

$(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$

initially by laboriously multiplying the previous result by another (1 + x).

But the coefficients can be determined directly from Pascal's triangle. All display wall charts start with a single "one" but that is wrong. It's two "ones" on the top row because it's the binomial and we have

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

So we can immediately write down the next expansion, without even bothering with those "combinations" like ${}^5C_2$ to find each term.

$(1 + x)^5 = 1+5x+10x^2+10x^3+5x^4+x^5$

What is less well known is the trinomial expansion and all its successive sisters, the quadrinomial, the quintinomial, the sextinomial and beyond for those who know their Latin.

Here's the trinomial

$(1 + x + x^2)^2 = 1+2x+3x^2+2x^3+x^4$

$(1 + x + x^2)^3 = 1+3x+6x^2+7x^3+6x^4 +3x^5+x^6$

$(1 + x + x^3)^4 = 1+4x+10x^2+16x^3 +19x^4+16x^5+10x^6+4x^7+x^8$

each term found by multiplying the previous expansion by $(1 + x + x^2)$

So is there a "super-Pascal" triangle that will give us these coefficients? Indeed there is, and this time we start with three "ones" and find successive rows by adding the three numbers above.

1 1 1

1 2 3 2 1

1 3 6 7 6 3 1

1 4 10 16 19 16 10 4 1

You might notice that the third row starts off like triangular numbers but then tails off because the row above also falls back. The fourth row similarly is the tetrahedral numbers. All these you can see in the diagonals of the original Pascal's triangle. So here there is a hint of how the normal distribution is an approximation of the binomial (and its higher sisters) in the limit. So now we can read off directly the coefficients for our next expansion.

The row is

1, 5, 15, 30, 35, 51, 35, 30, 15, 5, 1

and the expansion is

$$(1 + x + x^2)^5 = 1+5x+15x^2+30x^3+35x^4+ 51x^5+35x^6+30x^7+15x^8+5x^9+x^{10}$$

I'm going to pass over the quadrinomial and quintinomial and go straight to the sextinomial $(1+x+x^2+x^3+x^4+x^5+x^6)^6$ because that reflects the outcomes when you throw multiple dice. We build up the triangle in exactly the same way starting this time with six "1's". The only minor difference is that the numbers now go in the gaps just like the original triangle.

To simplify I'm no longer nesting and not repeating all the descending terms

1, 1, 1, 1, 1, 1

1, 2, 3, 4, 5, 6, 5, 4, 3, 2, 1

1, 3, 6, 10, 15, 21, 25, 27, 25, 21,15…

Now this third row allows us to read off directly the probability of scoring any total with three dice. The row sums to 216, which is 6^3, the total number of combinations. So there is a $^{1}/_{216}$ chance of scoring total "three", a $^{3}/_{216}$ chance of scoring total "four", all the way across until we have (again) a $^{1}/_{216}$ chance of scoring "eighteen"

Calulating by other methods is tricky.

For four dice the line starts

1,4,10,20,35,56,80,104,125,140,146,140

These are the number of combinations for totals "four", "five", "six" etc. So the probability of scoring a total of "eleven" with four dice is $^{104}/_{1296}$.

In 99.9% of cases when amateurs think they've discovered something, they have actually been beaten by someone else – in 90% of those cases by Euler[†].

Could this be that 0.1%?

† *63% of all statistics are made up.*

Goodhand's Trapezium

I've noticed a trend that on the back of funeral service sheets a piece of creative drawing or similar by the deceased may be printed. I'd like this please :-

Suppose I have a tetrahedral die – 4 sides marked '1', '2', '3' and '4'.

I write	'1'	'2'	'3'	'4'	Total
	1	1	1	1	4

so the probability of a total of say '2' in one throw is $^{1}/_{4}$. Not impressed?
Well write another row below adding up the four numbers above

°		°		1		1		1		1		°		°	Total
	1		2		3		4		3		2		1		16

and add a row above showing the possible totals of two dice

	'2'		'3'		'4'		'5'		'6'		'7'		'8'		Total
°		°		1		1		1		1		°		°	4
	1		2		3		4		3		2		1		16

so the probability of a total '4' in two throws is $^{3}/_{16}$. More impressed?

For three dice add another row, adding the 4 numbers above with a new header row.

'3'		'4'		'5'		'6'		'7'		'8'		'9'		'10'		'11'		'12'	Total
°		°		°		1		1		1		1		°		°		°	4
			1		2		3		4		3		2		1				16
1		3		6		10		12		12		10		6		3		1	64

so the probability of a total '6' in three throws is $^{10}/_{36}$.

Now you're impressed? Throwing 4 dice gives

'4'		'5'		'6'		'7'		'8'		'9'		'10'		'11'		'12'		'13'		'14'		'15'		'16'	Total
	°		°		°		°		1		1		1		1		°		°		°		°		4
						1		2		3		4		3		2		1							16
			1		3		6		10		12		12		10		6		3		1				64
1		4		10		20		31		40		44		40		31		20		10		4		1	256

so the probability of a total say '7' when throwing four dice is $^{20}/_{256}$.

Now you're really impressed. There are other patterns in these numbers – can you spot them? I would have liked to have shown you this table for a 6 sided die but this service sheet is too small to contain it#.

These tables took me about 4 years on and off to discover and came about after a lesson on dice throwing with a "bottom" set in Maths. There's always something new to discover in Maths.

Robert Goodhand

 # acknowledgments to Fermat