Mr G's Little Book on

Differentiating and Integrating 24 Trigonometric Functions

This booklet should contain an insert showing the domains of all 72 expressions

Differentiating Circular Functions

1) sin θ

$^{d}/_{d\theta}$ sin θ

= { sin(θ + δθ) – sin θ } / δθ

= { sinθ.cosδθ + cosθ.sinδθ – sinθ } / δθ

as δθ → **0** sin δθ → δθ and cos δθ → 1

= cos θ . δθ / δθ

= cos θ

2) cos θ

$^{d}/_{d\theta}$ cos θ

= { cos (θ + δθ) – cos θ } / δθ

= {cosθ.cosδθ – sinθ.sinδθ – cosθ} / δθ

= – sin θ . δθ. / δθ

= – sin θ

3) tan θ

$^{d}/_{d\theta}$ tan θ

= { cos θ.cos θ – sin θ.sin δθ } / $\cos^2$ θ

= $\sec^2$ θ or 1 + $\tan^2$ θ

4) cosecθ

$^{d}/_{d\theta}$ cosecθ

= $^{d}/_{d\theta}$ 1 / sinθ

= – cos θ / $\sin^2$ θ

= – cosec θ . cot θ

5) secθ

$^{d}/_{d\theta}$ sec θ

= $^{d}/_{d\theta}$ 1 / cos θ

= sin θ / $\cos^2$ θ

= sec θ . Tan θ

6) cot θ

$^{d}/_{d\theta}$ cot θ

= $^{d}/_{d\theta}$ cos θ / sin θ

= { – sinθ.sinθ – cosθ.cosθ } / $\sin^2$θ

= – cosec^2 θ

7) $\sin^{-1}$ x

Let y = $\sin^{-1}$ x

So x = sin y

∴ $^{dx}/_{dy}$ = cos y

= √ { 1 – $\sin^2$ y }

∴ $^{dy}/_{dx}$ = 1 / √ { 1 – x^2 }

8) $\cos^{-1}$ x

Let y = $\cos^{-1}$ x

So x = cos y

∴ $^{dx}/_{dy}$ = – sin y

= – √ {1 – $\cos^2$ y}

∴ $^{dy}/_{dx}$ = –1 / √ { 1 – x^2 }

9) $\tan^{-1} x$

Let $y = \tan^{-1} x$

So $x = \tan y$

$\therefore {}^{dx}/_{dy} = 1 + \tan^2 y$ *[from 3]*

$= 1 + x^2$

$\therefore {}^{dy}/_{dx} = 1 / (1 + x^2)$

10) $\text{cosec}^{-1} x$

Let $y = \text{cosec}^{-1} x$

So $x = \text{cosec } y$

$\therefore {}^{dx}/_{dy} = -\text{cosec } y . \cot y$ *[from 4]*

now $\text{cosec}^2 y = 1 + \cot^2 y$

So ${}^{dy}/_{dx} = -1 / \{ |x| \ \sqrt{(x^2 - 1)} \}$

11) $\sec^{-1} x$

Let $y = \sec^{-1} x$

So $x = \sec y$

$\therefore {}^{dx}/_{dy} = \sec y . \tan y$ *[from 5]*

now $\sec^2 y = 1 + \tan^2 y$

So ${}^{dy}/_{dx} = 1 / \{ |x| \ \sqrt{(x^2 - 1)} \}$

12) $\cot^{-1} x$

Let $y = \cot^{-1} x$

So $x = \cot y$

$\therefore {}^{dx}/_{dy} = -\text{cosec}^2 y.$ *[from 6]*

$= -(1 + \cot^2 y)$

So ${}^{dy}/_{dx} = -1 / (x^2 + 1)$

Differentiating Hyperbolics

13) sinh x

$\sinh x = x + x^3/3! + x^5 / 5! + x^7 / 7! \ldots$

So ${}^{d \sinh x}/_{dx} = 1 + x^2 / 2! + x^4 / 4! \ldots$

$= \cosh x$

14) cosh x

$\cosh x = 1 + x^2 / 2! + x^4 / 4!x + x^6 / 6! \ldots$

So ${}^{d \cosh x}/_{dx} = 1 + x^3 / 3! + x^5 / 5! \ldots$

$= \sinh x$

But that is not particularly enlightening so let's try an alternative way

$\cosh x = ½ (e^x + e^{-x})$

${}^{d \cosh x}/_{dx} = ½ (e^x - e^{-x})$

$= \sinh x$

15) tanh x

$\cosh x = 1 + x^2 / 2! + x^4 / 4!x + x^6 / 6! \ldots$

${}^{d \tanh x}/_{dx} = {}^{d}/_{dx} (\sinh x / \cosh x)$

$= (\cosh x.\cosh x - \sinh x.\sinh x) / \cosh^2 x$

remembering $\cosh^2 x - \sinh^2 x = 1$

hence $1 - \tanh^2 x = \text{sech}^2 x$

$= 1 - \tanh^2 x$

$= \text{sech}^2 x$

16) cosech x

${}^{d \text{ cosecht } x}/_{dx} = {}^{d}/_{dx} (1 / \sinh x)$

$= -\cosh x / \sinh^2 x$

$= -\text{cosech } x . \cot x$

17) sech x

$^{d\ secht\ x}/_{dx}$ = $^{d}/_{dx}$ (1 / cosh x)

= – sinh x / cosh² x

= – sech x . tanh x

18) coth x

$^{d\ cott\ x}/_{dx}$ = $^{d}/_{dx}$ (cosh x / sinh x)

= (sinh x.sinh x – cosh x.coshx) / sinh²x

= 1 – coth x

= – cosech² x

19) sinh⁻¹ x

Let y = $\sinh^{-1}$ x

So x = sinh y

∴ $^{dx}/_{dy}$ = cosh y

= √ (x^2 + 1)

∴ $^{dy}/_{dx}$ = 1 / √ (x^2 + 1)

20) $\cosh^{-1}$ x

Let y = $\cosh^{-1}$ x

So x = cosh y

∴ $^{dx}/_{dy}$ = sinh y

= √ (x^2 – 1)

∴ $^{dy}/_{dx}$ = 1 / √ (x^2 – 1)

21) $\tanh^{-1}$ x

Let y = $\tanh^{-1}$ x

So x = tanh y

∴ $^{dx}/_{dy}$ = sech^2 y *[from15]*

= (1 – $\tanh^2$ y)

∴ $^{dy}/_{dx}$ = 1 / (1 – x^2)

22) cosech^{-1} x

Let y = cosech^{-1} x

So x = cosech y

∴ $^{dx}/_{dy}$ = – cosech y . coth y *[from 4]*

now cosech^2 y = $\coth^2$ y – 1

∴ $^{dy}/_{dx}$ = –1 / { x √ (1 + x^2) }

23) sech^{-1} x

Let y = sech^{-1} x

So x = sech y

∴ $^{dx}/_{dy}$ = – sech y . tanh y *[from 17]*

now sech^2 y = 1 – $\tanh^2$ y

∴ $^{dy}/_{dx}$ = 1 / { x √ (1 – x^2) }

24) $\coth^{-1}$ x

Let y = $\coth^{-1}$ x

∴ x = coth y

∴ $^{dx}/_{dy}$ = – cosech^2 y. *[from 18]*

= 1 – $\coth^2$ y

∴ $^{dy}/_{dx}$ = 1 / (1 – x^2)

Integrating Circular Functions

25) sinθ

$\int \sin\theta \, d\theta = -\cos\theta + c$

26) cos θ dθ

$\int \cos\theta \, d\theta = \sin\theta + c$

27) tan θ

$\int \tan\theta \, d\theta = \int (\sin\theta / \cos\theta) \, d\theta$

Now from the log rule of differentiation

${}^{d}/_{dx}$ [ln f(x)] = f ′(x) / f(x)

$\rightarrow \quad = -\ln|\cos\theta| + c$

$\quad = \ln|\sec\theta| + c$

28) cosec θ

let $t = \tan \frac{1}{2}\theta$

${}^{dt}/_{d\theta} = \frac{1}{2}(1 + t^2)$ *[from 3]*

$d\theta = \{2 / (1 + t^2)\} \, dt$

Now here's the clever bit

because $\sec^2\theta = 1 + \tan^2\theta$

then $\sin 2\theta = (2\tan\theta) / (1 + \tan^2\theta)$

So $\sin\theta = \{2t / (1 + t^2)\}$

Putting this all together

$\int \text{cosec}\theta \, d\theta$

$= \int \{(1 + t^2) / 2t\} \times \{2 / (1 + t^2)\} \, dt$

$= \int 1 / t \, dt$

$= \ln(\tan \frac{1}{2}\theta)$

$= \ln|(\text{cosec}\,\theta - \cot\theta)|$

$-\ln|(\text{cosec}\,\theta + \cot\theta)| + c$

29) sec θ

$\int \sec\theta \, d\theta = \int 1 / \cos\theta \, d\theta$

$= \int (\cos\theta / \cos^2\theta) \, d\theta$

$= \int (\cos\theta / 1 - \sin^2\theta) \, d\theta$

Let $u = \sin\theta$ then $du = \cos\theta \, d\theta$

and our integral becomes

$\int du / (1 - u^2) = \tanh^{-1} u$ *[from 21]*

$\therefore \int \sec\theta \, d\theta = \tanh^{-1}(\sin\theta) + c$

The solution given in text books is more usually $\ln|\sec\theta + \tan\theta|$ *but these are equivalent*

30) cot θ

$\int \cot\theta \, d\theta = \int (\cos\theta / \sin\theta) \, d\theta$

which again is a standard form giving

$= \ln|(\sin\theta)| + c$

31) sin^{-1} x

$\int \sin^{-1} x \, dx = \int 1 \times \sin^{-1} x \, dx$

and integrating by parts give

$x \sin^{-1} x - \int \{x / \sqrt{(1 - x^2)}\}$

$= x \sin^{-1} x + \sqrt{(1 - x^2)} + c$

32) cos^{-1}x dx

$\int \cos^{-1} x \, dx = \int 1 \times \cos^{-1} x \, dx$

and integrating by parts give

$x \cos^{-1} x - \int \{-x / \sqrt{(1 - x^2)}\} \, dx$

$= x \cos^{-1} x - \sqrt{(1 - x^2)} + c$

33) $\tan^{-1} x$ dx

$\int \tan^{-1} x \, dx = \int 1 \times \tan^{-1} x \, dx$

and integrating by parts give

$x \tan^{-1} x - \int \{ x / \sqrt{(1 + x^2)} \} dx$

$= x \tan^{-1} x - \frac{1}{2} \ln (x^2 + 1) + c$

34) $\text{cosec}^{-1} x$ dx

$\int \text{cosec}^{-1} x \, dx = \int 1 \times \text{cosec}^{-1} x \, dx$

and integrating by parts give

$x \, \text{cosec}^{-1} x - \int \{ -x / x \sqrt{(x^2 - 1)} \} dx$

$= x \, \text{cosec}^{-1} x + \cosh^{-1} |x| + c$

35) $\sec^{-1} x$ dx

$\int \sec^{-1} x \, dx = \int 1 \times \sec^{-1} x \, dx$

and integrating by parts give

$x \sec^{-1} x - \int x / \{ x \sqrt{(x^2 - 1)} \} dx$

$= x \sec^{-1} x - \cosh^{-1} x + c$

36) $\cot^{-1} x$ dx

$\int \cot^{-1} x \, dx = \int 1 \times \cot^{-1} x \, dx$

and integrating by parts give

$x \cot^{-1} x - \int \{ -x / (1+x^2) \} dx$

$= x \cot^{-1} x + \frac{1}{2} \ln (x^2 + 1) + c$

Integrating Hyperbolics

37) sinhx dx

$\int \sinh x \, dx = \cosh x + c$

38) coshx dx

$\int \cosh x \, dx = \sinh x + c$

39) tanhx dx

$\int \tanh x \, dx = \int (\sinh x / \cosh x) \, dx + c$

$= \ln (\cosh x) + c$

40) cosech x

Taking our cue from $\int \text{cosec}\, x \, dx$

let $t = \tanh \frac{1}{2} x$

$dt/dx = \frac{1}{2} (1 - t^2)$ *[from 15]*

$dx = 2 / (1 - t^2) \, dt$

and by similar reasoning to [15}

$\sinh x = 2t / (1 - t^2)$

Putting this all together

$\int \text{cosech}\, x \, dx$

$= \int \{ (1 - t^2) / 2t \} \times \{ 2 / (1 - t^2) \} dt$

$= \int 1 / t \, dt = \ln (\tanh \frac{1}{2} x)$

$= \ln |(\coth x - \text{cosech}\, x)| + c$

41) sech x dx

Taking our cue from ∫ sec x dx

$\int \text{sech } x\, dx = \int (\cosh x / \cosh^2 x)\, dx$

$= \int \{ \cosh x / (1 + \sinh^2 x) \}\, dx$

Let u = sinh x

du = cosh x dx

$\int du / 1 + u^2 = \tan^{-1} u$ *[from 9]*

so $\int \text{sech } x\, dx = \tan^{-1} (\sinh x) + c$

42) coth x

$\int \coth x\, dx = \int (\cosh x / \sinh x)\, dx$

$= \ln (\sinh x) + c$

43) sinh^{-1}x **dx**

$\int \sin^{-1}x\, dx = \int 1 \times \sin^{-1}x\, dx$

and integrating by parts give

$x \sinh^{-1}x - \int \{x / \sqrt{\{x^2 + 1\}}$

$= x \sinh^{-1} x + \sqrt{(x^2 + 1)} + c$

44) cosh^{-1}x

$\int \cosh^{-1} x\, dx = \int 1 \times \cos^{-1}x\, dx$

and integrating by parts give

$x \cosh^{-1}x - \int x / \sqrt{(x^2 - 1)}\, dx$

$= x \cosh^{-1} x - \sqrt{(x^2 - 1)} + c$

45) tanh^{-1}x **dx**

$\int \tanh^{-1} x\, dx = \int 1 \times \tanh^{-1} x\, dx$

and integrating by parts give

$x \tanh^{-1} x - \int x / \sqrt{(1 - x^2)}\, dx$

$= x \tanh^{-1}x - \frac{1}{2} \ln (1 - x^2) + c$

46) cosech^{-1}x

$\int \text{cosech}^{-1} x\, dx = \int 1 \times \text{cosech}^{-1} x\, dx$

and integrating by parts give

$x\, \text{cosech}^{-1} x - \int \{-x / x \sqrt{(x^2 + 1)} \}dx$

$= x\, \text{cosech}^{-1} x + \sinh^{-1} x + c$

47) sech^{-1}x

$\int \text{sech}^{-1} x\, dx = \int 1 \times \text{sech}^{-1} x\, dx$

and integrating by parts give

$x\, \text{sech}^{-1}x - \int \{ -x / x \sqrt{(1 - x^2)} \}\, dx$

$= x\, \text{sech}^{-1} x - \cosh^{-1} x + c$

36) coth^{-1}x **dx**

$\int \coth^{-1} x\, dx = \int 1 \times \coth^{-1} x\, dx$

and integrating by parts give

$x \coth^{-1} x - \int \{ x / (1 - x^2) \}\, dx$

$= x \coth^{-1} x + \frac{1}{2} \ln |(1-x^2)| + c$

Footnote

In my first cut of this booklet I ignored all the principles of domain restriction. This did not become apparent until the last integral $\coth^{-1}x$ dx *where I could see there was a mismatch between the domains of the integrand and integral.*
I therefore mapped out the domains of all 72 expressions as shown on the insert and could see 5 domain issues. These I then corrected by inserting absolute values eg $|x|$ *into specific terms.*

Appendix

$\int \coth^{-1} x$

Derive gives the integral as

$\frac{1}{2}(x-\frac{1}{2})\ \ln(x+1)/\ln(x-1) - \ln(^1/_{x+1})$.

Now $x\coth^{-1}x + \frac{1}{2}\ln(x^2-1)$

$=\frac{1}{2}x[\ln(x+1)-\ln(x-1)]+$

$\frac{1}{2}[\ln(x+1)+\ln(x-1)$

$=\frac{1}{2}x\ln(x+1)- \frac{1}{2}\ln(x-1)- \frac{1}{2}x\ln(x-1)+$

$\frac{1}{2}\ln(x-1)+\ln(x+1)$

$= \frac{1}{2}x\ln(x+1)- \frac{1}{2}x\ln(x-1)+ \frac{1}{2}\ln(x+1)+$

$\frac{1}{2}\ln(x-1)-$

$= \frac{1}{2}x\ln(x+1)- \frac{1}{2}x\ln(x-1)+ \ln(x+1)-$

$\frac{1}{2}\ln(x+1)+ \frac{1}{2}\ln(x-1)$

$= \frac{1}{2}x\ln(x+1)- \frac{1}{2}\ln(x+1)- \frac{1}{2}x\ln(x-1)+$

$\frac{1}{2}\ln(x-1)+ \ln(x+1)$

$= \frac{1}{2}(x-1)[\ln(x+1)]- \frac{1}{2}(x-1)[\ln(x-1)]+$

$\ln(x+1)$

$= \frac{1}{2}(x-1)[\ln(x+1)-\ln(x-1)]+\ln(x+1)$

which matches above given last term is strangely inverted.

Alexa DOES give the integral as first derived.