Mr G's Little Book on

Conditional Probability

Conditional Probability

Conditional probability seems to be one of those subjects you sort of know about enough to teach the lower sixth but never actually get round to investigating its peculiarities for yourself. So this half term I'd thought I'd have a go. In the following narrative I admit up front I have absolutely no idea how the medical profession actually goes about its business. And I'm totally against pregnancy terminations!

I imagined a test for say Downs syndrome. I have no idea how such a test might be carried out but let's assume that eventuality someone is examining a pile of slides for abnormalities. There are two piles "the OK pile" and the "Looks dodgy" pile. If the slide is borderline it goes on the latter pile because there are other tests that can be carried out. So the "Looks dodgy" pile is likely to have more mistakes.

However the Chief Medical guy doesn't want too many women unnecessarily upset or expensive further tests being carried out for no good reason so he sets parameters. Let's say "**false positives**" no greater than 20%. As for false negatives and the preponderance for people to sue, he needs to keep that right down, say 1%. *(I might have made that even lower but you'll see later why I had to make it that high to see how the mathematics pans out)*

Now these conditional probabilities are not initially under his control, but let's use them anyway. For women actually carrying a Downs child let's say 99% are picked up by the test and for women not carrying a Downs child only 20% are unnecessarily passed up the chain.

Now there are numerous "proofs" for how to calculate conditional probability but to get an intuitive feel how about this?

The racehorse "Gödel's Revenge" has a ¼ chance winning if the going is soft. Checking the weather forecast, the punter determines the chance of soft going tomorrow is ½. So he reasons the chance of soft going tomorrow AND his horse winning is ½ x ¼ or better written ¼ x ½. But this is just the original conditional probability x the condition. So we have

P(A AND B) = P(A/B) x P(B) and hence

P(A/B) = P(A AND B) / P(B)

Now let's calculate the conditional probabilities for our Chief Medical guy.

First I look up on the Internet the incidence of Downs syndrome – 0.2% seems reasonable. So

P(Downs/+ve) = $^{\text{P(Downs AND +ve)}} / _{\text{P(+ve)}}$

P(Downs/-ve) = $^{\text{P(Downs AND –ve)}} / _{\text{P(-ve)}}$

P(Not D./+ve) = $^{\text{P(Not D. AND +ve}} / _{\text{P(+ve)}}$

P(Not D /-ve = $^{\text{P(Not D. AND -ve}} / _{\text{P(-ve)}}$

Now P(positive) =
0.002×0.99+0.998×0.2 =
0.20158

And P(negative) =
0.002×0.01+0.998×0.8 =
0.79842

which reassuringly add to one.

So P(Downs/test positive) =
0.002 x 0.99 / 0.201580 =
0.00982

P(Downs/test negative) =
0.002 x 0.01 / 0.798420 =
0.00003

P(Not D./test positive) =
0.998 x 0.2 / 0.201580 = 0.99018

P(Not D / test negative =
0.998 x 0.8 / 0.798420 = 0.99997

Here I've mercifully rounded off to 5 dp but the result is clear and our Chief Medical guy is tearing his hair out, assuming he has any. An amazing 99% of women who aren't carrying a Downs child are having further tests. At this rate we might as well abandon the test altogether and try something else. It's true he's had a fantastic success rate at making sure he's never sued but what's gone wrong?

Our med guy might have been good at Biology at school but should have paid more attention to his maths.

There's a world of difference between P(A/B) and P(B/A) particularly when the actual probabilities P(A) and P(B) are dissimilar. The heart of the problem lies in the fact that the actual incidence of Downs syndrome is less than the efficiencies of the test we've put in place to pick it up. And with the checker passing up all the borderline cases these two figures conspire to give us an unmanageable result.

I wrote a quick spreadsheet program so

I could enter values direct and see the results but even I was surprised at how high the levels of false positives came out. Setting both test parameters at 99.9% still results in an unbelievable 33% of false positives. I'm still thinking a coin wouldn't fall far behind.

Then I then looked up some figures on Wikipedia.

There is an alpha feto protein test and wiki quotes 79% detection rate and 7.5% false positives. That looks reassuringly low for false positives but hopeless on detection rates. I then thought that what I needed to do was redesign my spreadsheet so instead of entering the test parameters and calculating the conditional probabilities I could enter the conditional probabilities to find out the test parameters.

I'll spare the reader the lengthy algebra but given conditional probabilities x and y and incidence A, the two test parameters are given by

$T_1 = \{(1-x)/x-(1-A)/A\}/\{(1-x)/x-y/(1-y)\}$

and

$T_2 = ByA/\{(1-A)(1-y)\}$

These equations are believed original

I checked carefully that my new inverse machine was working correctly by feeding back in the answers from my first machine for several trials.

Finally all I had to do was feed in the wiki figures to discover how efficient the AFP test was. I was expecting some high figures but not (negative) 6129% for one. I just assumed that my machine would happily take any figures on conditional probability I dreamed up and produce the corresponding test parameters. In fact I discovered it was actually very difficult to find parameters that gave a sensible result and the best strategy was to nick them from machine one. I discovered there is no test that can give conditional probabilities 79% and 7.5% so either I was misreading them or they were in error.

But it did set me thinking. Suppose we translate all this investigation into the judicial system to the days when we had hanging, hence no chance of further testing the false positives. The condition now equates to "I did the murder", the test is the jury trial. Now assuming murderers are still relatively rare animals, could we reasonably feed in the same figures as our medical problem? That leaves us with the startling conclusion that about 99% of the people convicted and then hanged are likely innocent.

Can that seriously be true? Well unless we're confident that the efficiencies of the legal system are significantly superior to the actual incidence of murderers we're in danger of making the same mistake as our Chief Medical guy. The resolution is that it might be safe to assume that the proportion of people who actually turn up for trial is much higher than the incidence of Downs syndrome because a number of other "tests" have already been applied before the case ever comes to court, notwithstanding that people are innocent until proved guilty. But even in my lifetime names like Evans, Ellis and Bentley spring to mind.

rg