Finding Cubic Equations from Forward Difference Tables

This sheet is for students who understand fully the quadratic sheet

In any forward difference table, calculate each line of differences until you get a constant string. That tells you the order of the equation and the first coefficient.

One line	then it's linear	and the ?**n** is the last line $\div$ 1
Two lines	then it's a quadratic	and the ?$\mathbf{n}^2$ is the last line $\div$ (1x2) written 2!
Three lines	then it's a cubic	and the ?$\mathbf{n}^3$ is the last line $\div$ (1x2x3) written 3!
Four lines	then it's a quartic	and the ?$\mathbf{n}^4$ is the last line $\div$ (1x2x3x4) written 4!

Here's an example	***0***	***1***	***2***	***3***	***4***	***5***	***6***
		–4	**–4**	**10**	**50**	**128**	**256**

Construct the forward difference table and deduce the value at 0

0	***1***	***2***	***3***	***4***	***5***	***6***
–2	**–4**	**–4**	**10**	**50**	**128**	**256**
	–2	0	14	40	78	128
		2	14	26	38	50
			12	12	12	12

After three lines, the constant string is 12 so the first term is $^{12}/_{3!}$ = 2$\mathbf{n}^3$

Now take out that 2$\mathbf{n}^3$ value from the original set of results.

	0	***1***	***2***	***3***	***4***	***5***	***6***
Original Set Results	**–2**	**–4**	**–4**	**10**	**50**	**128**	**256**
Less 2$\mathbf{n}^3$	0	2	16	54	128	250	432
Gives	–2	–6	–20	–44	–78	–122	–176

Now being very careful on signs, carry on from the beginning with these new results.

0	***1***	***2***	***3***	***4***	***5***	***6***
–2	**–6**	**–20**	**–44**	**–78**	**–122**	**–176**
	–4	–14	–24	–34	–44	–54
		–10	–10	–10	–10	–10

Because we've taken out the 2$\mathbf{n}^3$ term, we're left with a quadratic or possibly even simpler if there is no n^2 term. Note the constant doesn't change either.

We can immediately deduce the other terms. The ?$\mathbf{n}^2$ term is half –10

Hence $-5n^2 + ?n - 2 = -6$ at n = 1 hence ? = –6 +5 + 2 = 1

and we have the final expression 2$\mathbf{n}^3$ – 5$\mathbf{n}^2$ + **n** – 2.

Here is another example, harder but more typical of the forward difference table you are likely to encounter in a real investigation.

Start with your results

0	***1***	***2***	***3***	***4***	***5***	***6***
	1	**2**	**4**	**9**	**19**	**36**

Construct the forward difference table and deduce the value at 0.

0	***1***	***2***	***3***	***4***	***5***	***6***
–1	**1**	**2**	**4**	**9**	**19**	**36**
	2	1	2	5	10	17
		–1	1	3	5	7
			2	2	2	2

After three lines, the constant string is 2 so the first term is $^{1}/_{3}$ $\mathbf{n}^3$ (because 2 ÷ (1x2x3) = $^{1}/_{3}$

Now take out that $^{1}/_{3}$ $\mathbf{n}^3$ value from the original set of results.

	0	***1***	***2***	***3***	***4***	***5***	***6***
Original Set Results	**1**	**1**	**2**	**4**	**9**	**19**	**36**
Less $^{1}/_{3}$ $\mathbf{n}^3$	0	$^{-1}/_{3}$	$^{-8}/_{3}$	$^{-27}/_{3}$	$^{-64}/_{3}$	$^{-125}/_{3}$	$^{-216}/_{3}$
Gives	–1	$^{2}/_{3}$	$^{-2}/_{3}$	$^{-15}/_{3}$	$^{-37}/_{3}$	$^{-68}/_{3}$	$^{-108}/_{3}$

Because we've now introduced a $^{1}/_{3}$ fraction, just work through the rest in thirds. Also be very careful on signs and carry on with these new results, which will now be a quadratic.

0	***1***	***2***	***3***	***4***	***5***	***6***
–1	$\mathbf{^{2}/_{3}}$	$\mathbf{^{-2}/_{3}}$	$\mathbf{^{-15}/_{3}}$	$\mathbf{^{-37}/_{3}}$	$\mathbf{^{-68}/_{3}}$	$\mathbf{^{-108}/_{3}}$
	$^{5}/_{3}$	$^{-4}/_{3}$	$^{-13}/_{3}$	$^{-22}/_{3}$	$^{-31}/_{3}$	$^{-40}/_{3}$
		–3	–3	–3	–3	–3

Because we've taken out the $^{1}/_{3}$ $\mathbf{n}^3$ term that we're left with a quadratic.

We can now deduce the other terms, though we must proceed with care.

The ?$\mathbf{n}^2$ term is half of (–3) ie ($^{-3}/_{2}$ $\mathbf{n}^2$). So at n = 1 we have

$^{-3}/_{2}$ + ? – 1 = $^{2}/_{3}$ so ? = $^{19}/_{6}$

Put the whole lot together and we have $^{1}/_{3}\,\mathbf{n}^3 - {}^{3}/_{2}\,\mathbf{n}^2 + {}^{19}/_{6}\,\mathbf{n} - 1$

Actually that's not as bad as it looks because if we take out a factor $^{1}/_{6}$ we get

$^{1}/_{6}$ $(2\mathbf{n}^3 - 9\mathbf{n}^2 + 19\mathbf{n} - 6)$ *(which doesn't look too bad.)*

Factors of $^{1}/_{3}$ and $^{1}/_{6}$ are very common in cubics *(why?)*