Mr G's Little Book on

Eigen Vectors

and

Eigen Values

Introduction

For summer 2010 I was seeking a suitable PDP topic. At the TSI summer school where I worked this year, students were free to choose a collective topic for lecture and investigation on the final day each week. Students were aged around 14 – 17 and preparing for GCSE final year. In the third week, students were adamant they wished to investigate matrices, having heard the topic but with no direct experience of them.

Studying textbooks, I realised at "A" level, matrices are only covered in FP3 with little explanation as to their properties and applications. IB Standard level covers a good introduction to matrices but then seems to digress into a lot of pointless "algebraic" manipulation without pursuing any higher level features that make matrices so important in engineering and physics.

I therefore designed and delivered a two hour "accessible"course which covered the basic features of matrices plus an explanation of their application in solving linear equations and their properties when viewed as transformation operators.

However I then realised my own appreciation and understanding of aspects such as eigenvalues and eigenvectors were purely procedural and decided to investigate these further.

I discovered there were no reliable TI-83 programs available on the internet to calculate eigenvalues and associated eigenvectors of an $n \times n$ matrix. There were a few limited applications available for 2×2 and 3×3 matrices.

An eigvl() function is available on the TI-nspire calculator but it appears to use a "trial and improvement" procedure and gave me no insight into the nature of eigenvalues and eigenvectors.

Investigating the mathematics underpinning eigenvalues and eigenvectors enabled me to write an algorithm covering any size square matrix. It further led me to the discovery an appreciation of exponential matrices.

Robert Goodhand

Mathematical Summary

Eigenvectors

If a square matrix [D] turns matrix **v** into a numerical multiple of itself then **v** is said to be an eigenvector of [D] with eigenvalue λ.

Eigenvectors correspond physically to a quantum state in which measuring an observable quantity [D] will with certainty give a result λ.

So we have $D\,v = \lambda\,v$ where λ is a scalar quantity.

$P(\lambda)$ is the characteristic polynomial and setting $P(\lambda) = 0$ gives us an n^{th} order polynomial in λ termed the "characteristic equation". The set of solutions are the eigenvalues.

The formal determination for any size square matrix is (starting with)

$$D\,v = \lambda\,v$$

Then to transform the scalar quantity λ into a vector by × by the identity matrix **I**

$$D\,v \quad = \lambda\,\mathbf{I}\,v$$

So $\qquad (D - \lambda\,\mathbf{I})\,v \quad = 0$

$(D - \lambda\,\mathbf{I})$ is termed the "coefficient matrix". We have, I effect produced a new matrix by deducting λ from the diagonal of the original matrix.

We then determine the characteristic equation by setting

$$\det(D - \lambda\mathbf{I}) = 0.$$

A quick way to determine eigenvalues for a 2×2 matrix is to use the product and sum relationships for the eigenvalues.

$$\lambda_1 \times \lambda_2 \quad = \det \quad [\,D\,] \quad \text{say} \quad \alpha$$

$$\lambda_1 + \lambda_1 \quad = \text{trace}\,[\,D\,] \quad \text{say} \quad \beta$$

which immediately gives the characteristic equation

$$\lambda^2 + {}^{-}(\alpha\beta)\,\lambda + (\alpha + \beta) = 0$$

To determine matrix [D]

Step 1

Suppose I postulate any three eigenvalues $\quad \lambda_1 = 2 \quad \lambda_2 = 4 \quad \lambda_3 = 5$

I enter these into a 3×3 diagonal matrix [B], the diagonal being $\lambda_1\ \lambda_2\ \lambda_3$.

Step 2

Next I specify three arbitrary eigenvectors as column vectors **v**.

$\lambda \quad 2 \quad 4 \quad 5$

v 3 –2 4

1 1 2

2 4 3

I enter these as the columns of a 3×3 matrix [A]

Step 3

I determine $[\ A\ ]^{-1} \Rightarrow [\ C\ ]$

Step 4

Now I determine matrix [D] from the relationship

$$[\ D\ ] = [\ P\ ]\,[\ J\ ]\,[\ P\ ]^{-1}$$

ie Matrix D with det[D] $\neq 0$ can be diagonalised into Jordan form [J] via a partner matrix [P].

So I enter $[\ P\ ]\,[\ J\ ]\,[\ P\ ]^{-1} \Rightarrow [\ D\ ]$

So starting with an arbitrary diagonal matrix [J], I have determined a matrix [D] that could be reduced to that diagonal form through another arbitrary partner matrix.

The elements of [D] are

–0.2 21.8 –7.6

–1.6 12.4 –2.9

–2.9 15.1 –1.2

(all to 1 dp) and I retain these in [D] for use later.

To determine Eigenvalues and E-vectors

Step 1

Enter into the TI–83 graph plotter

Y1 = det[D] – X×identity(3)

This deducts the as yet unknown eigenvalue λ from the matrix diagonal.

Studying the graph, I identify the x–intersects 2, 4 and 5, as expected.

Step 2

Taking $\lambda = 2$ enter ref [D] – 2 × identity (3) $\Rightarrow$ [E]

This deducts the known eigenvalue $\lambda = 2$ from the diagonal of a Gaussian reduction of the coefficient matrix by using the TI–83 row echelon form function.

Row 3 is zeros because there are infinitely many eigenvectors for this eigenvalue.

Step 3

I redimension [E] to an augmented 3×4 matrix, which adds a final column of zeros.

I now have a matrix [E] representing a set of linear equations that can be solved directly.

Step 4

I open Polysmlt Simultaneous equations and set dimensions to 4 × 4.

There appears to be a "bug" in Polysmlt, which deletes the final row when loading a matrix direct.

I load [E] and solve to get

$x_1 = 1.5x_3$

$x_2 = 0.5x_3$

$x_3 = x_3$

3

which matches the arbitrary column matrix eigenvector 1 originally chosen as the lowest integer solution. 2

I can then repeat this process for $\lambda = 4$ and $\lambda = 5$

Stage 4 Diagonalising Matrices, Eigenvalues and Eigenvectors

In the procedure for diagonalising any matrix [D] we seek to determine a partner matrix P and its inverse P^{-1} such that

$$[D] = [P] \times [J] \times [P]^{-1}$$

It can be shown that

1) the elements of diagonalised matrix *(Jordan form)* [J] are the eigenvalues of [D] and

2) the columns of partner matrix [P] are the associated eigenvectors.

which was an unexpected relationship and a serendipitous discovery for me.

If we wish to rearrange

$[D] = [P] \times [J] \times [P]^{-1}$ then we must proceed carefully

$[P]^{-1} \times [D] \times [P] = [P]^{-1} \times [P] \times [J] \times [P]^{-1} \times [P]$

so $[J] = [P]^{-1} \times [D] \times [P]$

Many websites get this back-to-front probably because the authors, while appreciating that you cannot rearrange matrix equations as if they were algebraic equations, then incorrectly assume that because $[P]^{-1}$ and [P] *are commutative they can be reversed with impunity!*

Extension Investigation - Exponential Matrices

If we define for matrix [D]

$$e^{[D]} = \mathbf{I} + [D] + [D]^2 \times (^1/_{2!}) + [D]^3 \times (^1/_{3!}) \ldots$$

where **I** is the identity matrix, it can be shown that

$$e^{[D]+[E]} = e^{[D]} \times e^{[E]}$$

providing [D] × [E] = [E] × [D]

The converse does not always hold, so this is not a "necessary condition".

If [J] is any diagonal matrix then it is easy to demonstrate that [$e^{[J]}$] is given by substituting each element "a" with "e^a".

This means that if we can successfully diagonalise the matrix [D] such that

$$[J] = [P]^{-1} \times [D] \times [P]$$

then as previously demonstrated

$$[D] = [P] \times [J] \times [P]^{-1}$$

so $a^{[D]} = [P] \times [J] \times [P]^{-1} [P] \times [J] \times [P]^{-1} \ldots [P] \times [J] \times [P]^{-1}$ *("a" times)* and eliminating "a" lots of $[P]^{-1} [P]$ and extending from "a" to "e", we get

$$e^{[D]} = [P] \times [e^{[J]}] \times [P]^{-1}$$

(again many websites get this back–to–front)

Finally it can be shown that

$$\det [e^{[D]}] = e^{\mathrm{trace}\,[D]}$$

a most esoteric and relatively unknown identity.

I wrote a TI-83 program to calculate [$e^{[D]}$] from the infinite series, pursuing terms until the next term made less than a predetermined absolute difference in the trace of [$e^{[D]}$] calculated to that point. The relationship $\det [e^{[D]}] = e^{\mathrm{trace}\,[D]}$ was then confirmed numerically.