The End of Mathematics

It's always exciting to read about the latest developments in Medicine or Physics but does anyone think the same happens in Mathematics. I suspect the majority think the whole subject was pretty well wrapped up by the ancient Greeks – most of the IGCSE textbooks would be comprehensible to an educated time-travelling Grecian

But take our two main Mathematical syllabuses here at Taunton School. We could be teaching "A" level Mathematics in D1 and in Standard Level IB next door in D2. Think of the simplest concept – the set **N** of counting numbers. Does that include "zero"? Well it's "No" if you're studying "A" level and "Yes" if your studying IB. That's because the two are built on different axiomatic systems – the assumed unprovable truths.

For "A" level, it's Peano's axioms – the first one stating, "one is the lowest number" and the next three constructing the counting numbers. IB is built on set theory, with the second axiom (not even the first) being "There is a null set". That's {} or ϕ. So how many sets have we now got? Answer "one" and we're off first base with the counting numbers out of thin air.

Peanos axioms are self evident, up to the fifth, when we get the axiom of induction. It's the axiom we use in those questions when we're given the answer and then asked to prove it. To me it always seemed to be a bit of a cheat. I mean if we have to be given the answer to start with – what's the point? Where did the answer come from? At university, when I was first shown how to solve differential equations, the lecturer said "Let's assume the solution is in the form Ae^{-bx}". So where did that come from? Later we were introduced to integrating factors but they also seemed to materialise out of nowhere. And so it goes on with summing series and many other areas.

At the turn of the twentieth century, set theory seemed to offer a whole new approach to mathematics – a new foundation based on more powerful if less intuitive axioms. However as Gottlob Frege was putting the final touches to a three-volume work, Bertrand Russell sent him a paradox of set theory – to do with sets being members of themselves. Frege wrote as an introduction to Volume 3 almost as it rolled off the press "A scientist can hardly meet with anything more undesirable than to have the foundation give way just as the work is finished."

Maybe Russell felt guilty because he and Whitehead then embarked on a monumental work "Principia Mathematica" in an attempt to prop up the increasingly shaky structure of Set Theory. It was heavy going and took 362 pages before they managed to prove 1+1=2 (this is not a joke). As at today we take the modified axioms of Zermelo-Fraenkel as our foundation. There is even a specific axiom, the axiom of regularity that disallows sets being members of themselves, the cause of the original problem.

But then there is that tricky Axiom of Choice (AC). Take set A consisting of three objects {a,b,c}, set B consisting of {d,e,f) and set C consisting of {g,h,j). The axiom of choice says we can form a new different set D by choosing an element from each of the three sets. Say D = {a,d,g}. There is no way to derive this self-evident fact from the other axioms. The main problem is the axiom of choice doesn't tell us how to choose the elements so many mathematicians try and work without it. Two in particular, Tarski and Banach wanted to banish it altogether. Their best attempt was to assume it true and then they set out to prove the most ridiculous thing they could – which was that if you cut up an orange into six (some say seven) pieces, you can rearrange the pieces and re-assemble them into an orange twice as big – with no gaps! Unfortunately the message the rest of the (mathematical) world got was – "wow, isn't maths amazing" and we carry on using the axiom of choice whenever necessary.

Having said all that, no one in practical everyday mathematics ever pays any of this the slightest notice though some issues are touched upon in Higher Level IB (eg proving two sets are equal)

Mathematicians are supposed to fall into different camps. First we have the logicists led by Frege and Russell who ultimately decamped. The latter eventually wrote in "Portraits from Memory" "Having constructed an elephant upon which the mathematical world would rest, I found the elephant tottering and proceeded to construct a tortoise to keep the elephant from falling. But the tortoise was no more secure than the elephant."

Then there are the constructivists led by Brouwer. He starts with the natural numbers as "fundamental intuition" and everything from thereon has to be constructed. The problem is constructivist mathematicians are prevented from using some of the everyday tools as a sort of "matter of principle". For example there is the law of trichotomy – "every real number is either positive negative or zero". No one would argue with that and it can be proved if the numbers are constructed set-theoretically. Not so for the constructivists who reject it. This is because the proof depends upon "proof by contradiction" which itself rests upon the law of the excluded middle. I'd have to say I've never met a constructivist – I think they are as rare as solipsists (though I am one of the latter).

Then among others there are the formalists, started by David Hilbert at the turn of the century. He put out a challenge to base mathematics on a complete and consistent set of axioms from which all truths could be automatically derived by following the rules. It would be like turning the handle of the machine in Swift's island of Laputa – whereby all works could ultimately be written by assembling the random lists of words produced.

Gödel put paid to that in 1931 with an amazing paper now termed the incompleteness theorem. The bottom line developed further by the British mathematician Alan Turing was that mathematics could <u>never</u> be both consistent and complete.

Imagine students attending the EFL. They are there to learn more fluent English. And what language do we use to achieve that? Why English! This is because the language itself and the individual's comprehension of the language exceeds some undefined critical mass such that English can be used to talk about itself meaningfully.

Now Gödel similarly proved that any sufficiently powerful axiomatic system could also talk about itself in the language of the system. Further no matter how many axioms are used as the foundation, it is possible to construct from those axioms a further statement that can be neither proved nor disproved within the system. This must then be appended as a further axiom and so the process continues.

Here's a flavour how he did it. He used "numbers" in a one-to-one correspondence with mathematical symbols so a number itself say 346,453,542,675,892 (I'm just making this number up) might be translated into some mathematical theorem, line by line, where the commas indicate a new line. The first really clever bit was working out a method so that part of the number say the "542" bit was an encoding of the whole number. There was a further clever bit using Cantor's diagonal argument. And the whole number said mathematically "This statement can not be proved within the system". But it could!

Finding actual examples is tricky because you're searching for the proof of something (say Fermat's Last Theorem that $a^n + b^n \neq c^n$ for n>2) never knowing if this happens to be something that cannot be proved. It's like searching the sock drawer for the missing sock when you don't even know if the sock is actually in the drawer or long thrown away. Fermat's last theorem was eventually proved true. That left the continuum hypothesis – how many points are there on a straight line. In fact two complementary proofs have been constructed which means there really are two different answers – termed C or $\square_1$. You might think that would make mathematics inconsistent. Not at all. It means you can adopted either as an additional axiom and it will not cause a contradiction in the whole system – assuming your original axioms are actually consistent of course. Philosophers are left to argue which of the two answers might really be true if you could step outside the system. Truth really does transcend proof.

There is a curious story about Gödel's application to become a citizen of the United States. He thought he had discovered a logical flaw in the American constitution whereby a dictatorship could arise as in Germany. His sponsors Einstein and Morgenstern urged him to steer well clear of the topic but at the interview the presiding judge actually asked Gödel about this. Before Gödel got into too deep water Morgenstern steered the conversation away from this sensitive area and Gödel was duly granted citizenship.

Let's finish with Gödel's second incompleteness theorem. "No axiomatic system can prove its own consistency". So for starters we cannot ever know if our axioms actually are consistent. More problematic, if you have an inconsistent axiomatic system you can actually prove anything – from 1 + 1 = 3 upwards. That means if in our presently constructed mathematics we manage to prove it is consistent, that means it's inconsistent – because only in an inconsistent system can we prove consistency. That really is Catch 22.

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Some themes in this article are taken from "Riddles in Mathematics" by E. Northrop, "Escher Gödel Bach" by D. Hofstadter and "The Mathematical Experience" by Davis and Hersh.