Exponential Functions

Assume there exists a function EXP(x)

such that $^{d}/_{dx}$ EXP(x) = EXP (x)

that is the function is its own derivative.

Now assume it can be expressed as a

power series EXP(x) = $a_0 + a_1x + a_2x^2 \ldots$

Now it is clear that each term must itself

be the derivative of the succeeding term.

So for simplicity set $a_0 = 1$.

Hence $a_1 = x$, $a_2 = {}^{x^2}/_{2!}$, $a_3 = {}^{x^3}/_{3!}$ etc. so

EXP(x) = $1 + x + {}^{x^2}/_{2!} + {}^{x^3}/_{3!} \ldots$

setting x= 1 we have

EXP(1) = $1 + 1 + {}^{1}/_{2!} + {}^{1}/_{3!} \ldots$

Assuming the series to be convergent we

denote the sum by e so EXP(1) = e

and in passing EXP(0) = 1

We need to demonstrated EXP(x) = e^x

The first step is to demonstrate that

EXP(a) + EXP(b) = EXP (a + b)

so we multiply out two infinite series

$(1+a+{}^{a^2}/_{2!}+{}^{a^3}/_{3!} \ldots) \times (1+b+{}^{b^2}/_{2!}+{}^{b^3}/_{3!} \ldots)$

Now you can either take it on trust or

actually embark on this. Whichever when

you write out all the terms in appropriate

columns you will find within each factorial

group the binomial terms appearing giving

for sure $1 + (a+b) + {}^{1}/_{2!}(a+b)^2 + {}^{1}/_{3!}(a+b)^3$

Hence EXP(a) + EXP(b) = EXP (a + b)

So now we are the first step in the

journey to show EXP is indeed an

exponential function.

Using the same technique of multiplying

out infinite series the result may be

extended to

EXP(a) × EXP(b) × EXP(c) = EXP (a+b+c)

Hence $[\text{EXP}(a)]^x$ = EXP (ax)

setting a = 1

$[\text{EXP}(1)]^x$ = EXP (x)

but we have already set EXP(1) = e

so now we have shown *(roll of drums)*

EXP(x) = e^x. and further

$e^x \times e^y = e^{x+y}$

This process is valid at this stage for x an

integer but by similar processes we can

extend into x rational, x negative and

ultimately x real.

Now we extend into the complex plane.

Let $y = \cos\theta + i \sin\theta$

$^{dy}/_{d\theta} = -\sin\theta + i\cos\theta$

$^{dy}/_{d\theta} = iy$

$\int {}^{dy}/_{y} = \int i d\theta$

In $y = i\theta + k$ and easily

At $\theta = 0$, $y = 1$ Iny = 0 hence k=0

Hence $y = e^{i\theta}$ and hence

$e^{i\theta} = \cos\theta + i\sin\theta$

Setting $\theta = \pi$ we get $e^{i\pi} = -1$

which is the neatest proof you've ever

seen *rg*

Footnote

It's relatively easy to demonstrate that

$(1 + {}^{1}/_{n})^{n} = e$ as $n \to \infty$

Simply expand as a binomial

$1 + n({}^{1}/_{n}) + n(n - 1)({}^{1}/_{n})^{2}({}^{1}/_{2!})$

$\quad + n(n - 1)(n - 2)({}^{1}/_{n})^{3}({}^{1}/_{3!})\ldots$

Even at this stage it's clear for n large the n terms on the top neatly cancel the $({}^{1}/_{n})$ terms but let's be a little more rigid.

$(1 + {}^{1}/_{n})^{n} = 1 + 1 + n^{2}(1-{}^{1}/_{n})({}^{1}/_{n})^{2}({}^{1}/_{2!})$

$\quad + n^{3}(1-{}^{1}/_{n})(1-{}^{2}/_{n})({}^{1}/_{n})^{3}({}^{1}/_{3!})$

as $n \to \infty$ $1/x \to 0$ so

$(1 + {}^{1}/_{n})^{n} = 1 + 1 + ({}^{1}/_{2!}) + ({}^{1}/_{3!}) \ldots$

which is our original expression for EXP(n)