Forward Difference Tables and Falling Factorials

Summary

The first line of integer values of a polynomial f(x) degree n produces a second line of difference values given by g(x). The Δ operator is defined such that $g(x) = \Delta f(x)$ and a simple proof given why $\Delta^n x^n = n!$

While g(x) may be determined by algebraic manipulation this paper introduces falling factorials $x^{[n]}$ such that $\Delta x^{[n]} = nx^{[x-1]}$. Sterling numbers S_1 and S_2 convert the expression x^n and lowers degrees to $x^{[n]}$ enabling direct discrete differentiation to be applied to determine the next line of the forward difference table and then to restore the polynomial to standard form.

The Δ and $x^{[n]}$ operators are then used to determine f(x) directly from a given set of data points.

In conclusion this paper demonstrates the forward difference operator Δ as discrete differentiation and its connection with Taylor's Series.

A procedure to determine the simplest polynomial of degree n from n discrete values is defined.

Introduction

It is generally appreciated that

For $f(x) = x$ finite differences are

0 1 2 3 4 5

1 1 1 1 1 (ie 0!)

For $f(x) = x^2$ finite differences are

0 1 4 9 16 25

1 3 5 7 9

2 2 2 2 (ie 1!)

For $f(x) = x^3$ finite differences are

0 1 8 27 64 125

1 7 19 37 61

6 12 18 24

6 6 6 (ie 2!)

For $f(x) = x^4$ finite differences are

0 1 16 81 256 625

1 15 65 175 369

14 50 110 194

36 60 84

24 24 (ie 3!)

For $f(x) = x^5$ finite differences are

0 1 32 243 1024 3125

1 31 211 781 2101

30 180 570 1320

150 390 750

240 360

120 (ie 4!)

but a simple intuitive explanation is rarely forthcoming.

Show by induction $\Delta_n x^n = n!$

where $\Delta_1 = f(x+1) - f(x)$

and Δ_k is the k^{th} finite difference which may not be a constant.

Now clearly $\Delta_1 x^1 = 1!$

Assume $\Delta_n x^n = n!$

$$\int \Delta^n x^n \, dx = \int n! \, dx$$

Assuming Δ^n to be a constant

$(\Delta^n x^{n+1}) / (n+1) + C_1 = n! \, x + C_2$

setting $x = 0$ demonstrates $C_1 = C_2$ so

$$\Delta^n x^{n+1} = (n+1)! \, x$$

Now take the next finite difference

$$\Delta^{n+1} x^{n+1} = \Delta_1 (n+1)! \, x$$

Going back to the definition of Δ_1

$\Delta^{n+1} x^{n+1} = (n+1)! \, (x+1) - (n+1)! \, x$

and multiplying out rhs gives

$$\Delta^{n+1} x^{n+1} = (n+1)!$$

Hence by induction $\boldsymbol{\Delta_n x^n = n!}$

Example

Let $f(x) = x^2+2x+1$

$\Delta f(n) = 4(x+1) + 2 - (4x + 2) = 4$

as expected.

However it becomes extremely tedious to calculate every line as a polynomial for higher degrees. Hence we introduce the falling factorial.

Falling Factorial Approach

Define $x^{[n]} = x(x-1)(x-2)\ldots(x-n+1)$

that is n terms.

$x^{[n]} = {}_{k=0}\prod^{n-1}(x-k) = n!\ {}^{x}c_n$

so

$x^{[1]} = x$

$x^{[2]} = x(x-1)$

$x^{[3]} = x(x-1)(x-2)$ etc.

and we define

$x^{[0]} = 1$ the empty product.

For reference the rising factorial is

$x^{(n)} = x(x+1)(x+2)\ldots(x+n-1)$

$= {}_{k=0}\prod^{n-1}(x+k)$

There is no universal agreement on the notation. The author fixes $x^{[n]}$ for the falling factorial and $x^{(n)}$ for the rising factorial as the most logical compromise of competing notations.

Now we can determine

$\Delta x^{[n]} =$

$(x+1)(x)(x-1)(x-2)\ldots(x-n+3)(x-n+2)$

minus

$(x)(x-1)(x-2)\ldots(x-n+2)(x-n+1) =$

$\{(x)(x-1)(x-2)\ldots(x-n+2)\} \times$

$\{(x+1)-(x-n+1)\}$

which conveniently simplifies to

$\boldsymbol{\Delta x^{[n]} = nx^{[n-1]}}$

because the last term is missing

and we compare to ${}^{d}/_{dx}\, x^n = nx^{n-1}$

Example

Now define $\Delta x^{[3]} = (x+1)^{[3]} - x^{[3]}$

$= (x+1)(x)(x-1) - x(x-1)(x-2)$

which if we multiply all out reduces to

$3x(x-1)$ which is $3x^{[2]}$

Now if we take the polynomial

$f(x) = ax^3 + bx^2 + cx + d$ we write

$f(x) = A[x]^3+B[x]^2+C[x] + D$

where A B C D are constants and different from a b c d but may be expressed in terms of a b c d if we bothered to multiply out.

We may thus show

$\Delta x^{[1]} = 1$

$\Delta x^{[2]} = 2x$

$\Delta x^{[3]} = 3x^{[2]}$

$\Delta x^{[4]} = 4x^{[3]}$

So the discrete Δ function representing the forward difference recreates the differential function when we express x^n in $x^{[n]}$ terms.

Hence starting with say

$Ax^{[3]} + Bx^{[2]} + Cx + D$

the $\Delta_3\{f(x)\} = 6a$

because Δ_3 of the other terms will all reduce to zero. This explains the original premise but does not yet give a method for switching between a b c d and A B C D. For this we need Stirling Numbers.

Stirling Numbers of the 1st Kind S_1

These are used to express falling factorials in terms of powers ox x

$x^{[1]} = x$

$x^{[2]} = x(x-1)$

$= x^2 - x$

$x^{[3]} = x(x-1)(x-2)$

$= x^3-3x^2+2x$

$x^{[4]} = x(x-1)(x-2)\ (x-3)$

$= x^4-6x^3+11x^2-6x$

$x^{[5]} = x(x-1)(x-2)\ (x-3)\ (x-4)$

$x^5-10x^4+35x^3-50x^2+24x$

Tables are available for higher polynomials.

The highest power is always one.

Stirling Numbers of the 2nd Kind S_2

These are used to express the powers of x in terms of falling factorials.

$x = x[1]$

$x^2 = (x^2-x) + x$

$= x^{[2]} + x^{[1]}$

ie you enter the first Stirling expression and then balance off each successive term in sequence

$x^3 = x^3-3x^2+2x)+3(x^2-x)+x$

$= x^{[3]} + 3x^{[2]} + x^{[1]}$

$x^4 = (x^4-6x^3+11x^2-6x) +$

$6(x^3-3x^2+2x)+7(x^2-x)+x$

$= x^{[4]}+6x^{[3]}+7x^{[2]}+x^{[1]}$

$x^5 = (x^5-10x^4+25x^3-15x^2+x)$

$+10(x^4-6x^3+11x^2-6x)$

$+25(x^3-3x^2+2x) +15(x^2-x)+x$

$= x^{[5]}+10x^{[4]}+25x^{[3]}+15x^{[2]}+x^{[1]}$

Tables are available for higher polynomials.

The lowest power is always one.

Example

Let $f(x) = 3x^5-6x^4+2x^3-4x^2+5x-1$

We translate into falling factorial notation by using S_2.

	$x^{[5]}$	$x^{[4]}$	$x^{[3]}$	$x^{[2]}$	$x^{[1]}$	k
$3x^5$	3	30	75	45	3	
$^-6x^4$		$^-6$	$^-36$	$^-42$	$^-6$	
2x3			2	6	2	
4x2				−4	−4	
5x−1					5	−1

giving f(x) =

$3x^{[5]}+24x^{[4]}+41x^{[3]}+5x^{[2]}+0x^{[1]} -1$

Hence $\Delta f(x)$ =

$15x^{[4]}+96x^{[3]}+123x^{[2]}+10x^{[1]}$

Now we translate that back into polynomial notation by using S_1.

	x^4	x^3	x^2	x^1
$15x^{[4]}$	15	$^-90$	165	$^-90$
$96x^{[3]}$		96	288	192
$123x^{[2]}$			123	$^-123$
$10x^{[1]}$				10
	15	6	0	$^-11$

Hence the first row of differences are given by $g(x) = 15x^4+6x^3+0x^2-11x$

Checking this result

Using a graphical calculator determine the first few values of

$3x^5-6x^4+2x^3-4x^2+5x-1$

x	0	1	2	3	4	5
	$^-1$	$^-1$	9	275	1619	5799
1st diff	0	10	266	1344	4180	

Now setting $g(x) = 15x^4+6x^3-11x$ does give coefficients 0,10, 266, 1344, 4180

As a final check we can enter into the graphical calculator the whole expression for Δ ie

$\{3(x+1)^5-6(x+1)^4+2(x+1)^3-4(x+1)^2+5(x+1)-1\} - \{3x^5-6x^4+2x^3-4x^2+5x-1\}$

which indeed does give the 1st difference coefficients

0,10, 266, 1344, 4180

Determining f(x) from a sequence

x	0	1	2	3	4	5
	f(0)	f(1)	f(2)	f(3)	f(4)	f(5)
		Δf(0)	Δf(1)	Δf(2)	Δf(3)	Δf(4)
			$\Delta^2f(0)$	$\Delta^2f(1)$	$\Delta^2f(2)$	$\Delta^2f(3)$
				$\Delta^3f(0)$	$\Delta^3f(1)$	$\Delta^3f(2)$ $\Delta^3f(3)$

Now suppose we have the polynomial

$f(x) = a_0 + a_1x + a_2x^2 \ldots a_{n-1}x^{n-1} + a_nx^n$

but then we transform this polynomial using S_2 into terms of $x^{[n]}$ with coefficients b_n as yet undetermined.

$f(x) = b_0+b_1x^{[1]}+b_2x^{[2]}\ldots b_{n-1}x^{[n-1]}+b_nx^{[n]}$

Now we apply the rule

$\Delta x^{[n]} = nx^{[n-1]}$ repeatedly

$\Delta f(x) = b_1+2b_2x^{[1]}+3b_3x^{[2]}+4b_4x^{[3]} \ldots$

$\Delta^2f(x) = 2!b_2x^{[1]}+(3\times2)b_3x^{[1]}+(4\times3)b_4x^{[2]}\ldots$

$\Delta^3f(x) =$

$3!b_3x+(4\times3\times2)b_4x^{[1]}+(5\times4\times3)b_5x^{[2]}\ldots$

$\Delta^4f(x) =$

$4!b_4x+(5\times4\times3\times2)b_5x^{[1]}+(6\times5\times4\times3)b_6x^{[1]}$

Now set x=0 and all terms disappear except the first

$f(0) = b_0$ hence $b_0 = f(0)$

$\Delta f(0) = b_1$ hence $b_1 = \Delta f(0)$

$\Delta^2f(0) = 2!\, b_2$ hence $b_2 = \Delta^2f(0) / 2!$

$\Delta^3f(0) = 3!\, b_3$ hence $b_3 = \Delta^3f(0) / 3!$

$\Delta^4f(0) = 4!\, b_3$ hence $b_4 = \Delta^4f(0) / 3!$

Hence

$f(x) = f(0)+ \Delta f(0)x^{[1]}+ \Delta^2f(0) / 2!\, x^{[2]}$ etc

That is we construct the forward difference table and just take the leading terms of each row multiplied by the falling factorial divided by n!

Example

x	0	1	2	3	4	5
f(x)	¯2	¯4	¯4	10	50	128
		¯2	0	14	40	78
		2	14	26	38	
			12	12	12	

Therefore we determine we have a polynomial of the third degree

Let $f(x) = ax^3 + bx^2 + cx + d$

$f(x) = f(0) + \Delta f(0)x^{[1]} + \Delta^2 f(0) / 2!\ x^{[2]}$

$f(x) = {}^-2 + {}^-2\ x + {}^2/_{2!}\ x(x-1)$

$\quad + {}^{12}/_{3!}\ x(x-1)\ (x-2)$

$= -2+(-2-1+4)x+(1-6)x^2+(2)x^3$

$= 2x^3 - 5x^2 + x - 2$

However there is a more intuitive way of determining the polynomial. The forward difference table immediately gives us the x^3 term ${}^{12}/_{3!} = 2$

Deduct this term from the original line

x	0	1	2	3	4	5
f(x)	¯2	¯4	¯4	10	50	128
$2x^3$	0	2	16	54	128	250
g(x)	¯2	¯6	¯20	¯44	¯78	¯122

and repeat the process

g(x)	¯2	¯6	¯20	¯44	¯78	¯122
		¯4	¯14	¯24	¯34	¯44
		10	10	10	10	

So now we have the x^2 term ${}^{10}/_{2!} = 5$

We now have

$f(x) = 2x^3 - 5x^2 + cx - 2$

given $f(1) = -4 = (a + b + c + d)$ then

$f(x) = 2x^3 - 5x^2 + x - 2$

Summary

Taking any function f(x) assumed a polynomial and construct the forward difference table until the constant is found say 6a indicating a cubic.

Calculate $f(x)^3$ and deduct this from the original series and repeat the process to determine b and finally c.

d is immediately given by f(0)

McLaurin Series

Note the McLaurin series for f(x) is

$f(0) + f'(0)\ x + {}^{f''(0)}/_{2!}\ x^2 + {}^{f'''(0)}/_{3!}\ x^3 + \ldots$

So the Δ function is just the discrete version of the differential function and the McLaurin series just reflects the pattern found from the discrete investigation.

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Acknowledgment is given to the paper "Formula Development through Finite Differences" by Brother Alfred Brousseau for the clear application of falling factorials. This seems the only publication on the internet that clearly covers this process.