Fibonnacci Properties – Paper1

Summary

It is an interesting property that for any Fibonnacci number F_k then

$F_{nk} \bmod F_k = 0$

that is every multiple of k in F_k is divisible by F_k. Here is a summary to F_{20}.

$F_0 = 0$

$F_1 = 1$ and obviously applies for F_1.

$F_2 = 1$	index prime so n/a	
$F_3 = 2$	index prime so n/a	
$F_4 = 3$	$3 \times F_2$	
$F_5 = 5$	index prime so n/a	
$F_6 = 8$	$4 \times F_3$	
$F_7 = 13$	index prime so n/a	
$F_8 = 21$	$7 \times F_4$	
$F_9 = 34$	$17 \times F_3$	
$F_{10} = 55$	$11 \times F_5$	
$F_{11} = 89$	index prime so n/a	
$F_{12} = 144$	$72 \times F_3$	$18 \times F_6$
$F_{13} = 233$	index prime so n/a	
$F_{14} = 377$	$29 \times F_7$	
$F_{15} = 610$	$305 \times F_3$	$122 \times F_5$
$F_{16} = 1597$	$47 \times F_8$	
$F_{17} = 1597$	index prime so n/a	
$F_{18} = 2584$	$1292 \times F_3$	$323 \times F_6$
	$76 \times F_9$	
$F_{19} = 4181$	index prime so n/a	
$F_{20} = 6765$	$1353 \times F_5$	$123 \times F_{10}$

Initial Investigations

$\mathbf{F_{2n+2} = F_{2n+1} + F_{2n+0}}$

$F_{3n+3} = F_{3n+2} + F_{3n+1}$

$= F_{3n+1} + F_{3n+0} + F_{3n+1}$

$\mathbf{F_{3n+3} = 2\,F_{3n+1} + 1\,F_{3n+0}}$

$F_{4n+4} = F_{4n+3} + F_{4n+2}$

$= F_{4n+2} + F_{4n+1} + F_{4n+1} + F_{4n+0}$

$= F_{4n+1} + F_{4n+0} + 2\,F_{4n+1} + F_{4n+0}$

$\mathbf{F_{4n+4} = 3\,F_{4n+1} + 2\,F_{4n+0}}$

$F_{5n+5} = F_{5n+4} + F_{5n+3}$

$= F_{5n+3} + F_{5n+2} + F_{5n+2} + F_{5n+1}$

$= F_{5n+3} + 2\,F_{5n+2} + F_{5n+1}$

$= F_{5n+2} + F_{5n+1} + 2\,F_{5n+1} + 2\,F_{5n+0} + F_{5n+1}$

$= F_{5n+2} + 4\,F_{5n+1} + 2\,F_{5n+0}$

$= F_{5n+1} + F_{5n+0} + 4\,F_{5n+1} + 2\,F_{5n+0}$

$\mathbf{F_{5n+5} = 5\,F_{5n+1} + 3\,F_{5n+0}}$

$F_{6n+6} = F_{6n+5} + F_{6n+4}$

$= F_{6n+4} + F_{6n+3} + F_{6n+3} + F_{6n+2}$

$= F_{6n+4} + 2\,F_{6n+3} + F_{6n+2}$

$= F_{6n+3} + F_{6n+2} + 2\,(F_{6n+2} + 2\,F_{6n+1}) + F_{6n+1} + F_{6n+0}$

$= F_{6n+3} + 3\,F_{6n+2} + 3\,F_{6n+1} + F_{6n+0}$

$= F_{6n+2} + F_{6n+1} + 3\,(F_{6n+1} + F_{6n+0}) + 3\,F_{6n+1} + F_{6n+0}$

$= F_{6n+2} + 7\,F_{6n+1} + 4\,F_{6n+0}$

$= F_{6n+1} + F_{6n+0} + 7\,F_{6n+1} + 4\,F_{6n+0}$

$\mathbf{F_{6n+6} = 8\,F_{6n+1} + 5\,F_{6n+0}}$

Hypothesis

The first hypothesis is that clearly when we decompose F_{kn+k} we will end up with

$A \times F_{kn+1} + B \times F_{kn+0}$

Now setting k = 0 gives

$F_k = A \times F_1 + B \times F_0$

Now as $F_1 = 1$ and $F_0 = 0$ clearly

$F_k = k \times F_1 + B \times F_0$

Here the value of B is immaterial

Nevertheless our second hypothesis is that $B = F_{k-1}$

We therefore predict $F_7 = 13F_1 + 8F_0$ which is immediately obvious

but setting n = 1

$F_{14} = 13\ F_8 + 8\ F_7$

$F_{14} = 13 \times 21 + 8 \times 13 = 377$

a true statement.

and setting n = 2

$F_{21} = 13\ F_{15} + 8\ F_{14}$

$F_{14} = 13 \times 610 + 8 \times 377 = 10946$

a true statement

Formal Proof

$F_{kn+k} = A \times F_{kn+1} + B \times F_{kn+0}$

setting n = 0

$F_k = A \times F_1 + B \times F_0$

hence A = k

setting n = 1

$F_{2k} = k \times F_{k+1} + B \times F_k$

hence $F_{2k} \bmod F_k = 0$

Irrespective of the value of B

Hence by induction

$F_{nk} \bmod F_k = 0$ QED

Narrative

If we decompose say F_{10} we can see clearly what's happening

$F_{10} = F_9 + F_8$

$= F_8 + F_7 + F_7 + F_6$

$= \mathbf{1}\ F_8 + \mathbf{2}\ F_7 + \mathbf{1}\ F_6$

$= F_7 + F_6 + 2\ (F_6 + F_5) + F_5 + F_4$

$= \mathbf{1}\ F_7 + \mathbf{3}\ F_6 + \mathbf{3}\ F_5 + \mathbf{1} F_4$

$= F_6 + F_5 + 3\ (F_5 + F_4) + 3\ (F_4 + F_3)$
$+ F_3 + F_2$

$= \mathbf{1}\ F_6 + \mathbf{4}\ F_5 + \mathbf{6}\ F_4 + \mathbf{4}\ F_3 + \mathbf{1}\ F_2$

$= F_5 + F_4 + 4\ (F_4 + F_3)$
$+ 6\ (F_3 + F_2) + 4\ (F_2 + F_1) + F_1 + F_0$

$= \mathbf{1}\ F_5 + \mathbf{5} F_4 + \mathbf{10} F_3 + \mathbf{10} F_2 + \mathbf{5} F_1 + \mathbf{1}\ F_0$

and up to this point we are producing the terms of Pascals Triangle. But in the next decomposition the last two terms $F_1$and F_0 do not decompose further.

$F_{10} = F_4 + F_3 + 5(F_3 + F_2) + 10(F_2 + F_1)$
$10(F_1 + F_0) + 5F_1 + F_0$

$= F_4 + 6\ F_3 + 15\ F_2 + 25\ F_1 + 11\ F_1$

and we continue with

$F_{10} = F_3 + 7\ F_2 + 46\ F_1 + 26\ F_0$

$F_{10} = F_2 + 54\ F_1 + 33\ F_0$

$F_{10} = 55\ F_1 + 34\ F_0$ as expected

Thus the number of coefficients of all the terms double at each decomposition until the end terms buffer up to F_1 and F_0 and thereafter pile up until we have just F_1 and F_0 remaining.

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Pascals and Fibonnacci

The conclusion of the investigation must be that Fibonnacci numbers relate to Pascal's triangle. To locate we have the terms that actually built up the final F_{10} coefficient and here they are

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1

Numbers on the diagonal lines add to successive Fibonnacci numbers.

Row Expansion

We can chart the number of terms in successive row of the expansion of F_n. and it is better to split them into odd and even rows

2n 2
4n 2 4 5
6n 2 4 8 12 13
8n 2 4 8 16 27 33 34
10n 2 4 8 16 32 58 80 88 89

3n 2 3
5n 2 4 7 8
7n 2 4 8 15 20 21
9n 2 4 8 16 31 47 54 55
11n 2 4 8 16 32 63 105 134 143 144

It is left as an exercise for the reader to find the general term for odd and even rows.