General Solutions from the Forward Difference Table

There is a precise mathematical way to produce the formula from the forward difference table, though you need a good command of algebra. Often you can spot a pattern from the simpler rules given previously and a bit of guesswork and trial and error. For those that want to take it further –here it is.

Let	n =	0	1	2	3	4	5	6
	f(n) =	a_0	a_1	a_2	a_3	a_4	a_5	a_6
			b_0	b_1	b_2	b_3	b_4	b_5
				c_0	c_1	c_2	c_3	c_4
					d_0	d_1	d_2	d_3
						e_0	e_1	e_2 etc.

Let $a = a_0/0!$ Let $b = b_0/1!$ Let $c = c_0/2!$ Let $d = d_0/3!$ etc. *(nb 0! = 1 by definition)*

1^{st} order occur when $b_0 = b_1 = b_2$ etc.

2^{nd} order occur when $c_0 = c_1 = c_2$ etc.

3^{rd} order occur when $d_0 = d_1 = d_2$ etc.

1^{st} order $f(n) = a + b\,n$

2^{nd} order $f(n) = a + b\,n + c\,n\,(n-1)$

$= a + (b - c)\,n + c\,n^2$

3^{rd} order $f(n) = a + b\,n + c\,n\,(n-1) + d\,n\,(n-1)\,(n-2)$

$= a + (b - c + 2d)\,n + (c - 3d)\,n^2 + d\,n^3$

which is about as complicated as you're ever likely to meet. *rg*

Producing Summations from the Forward Difference Table

There is a precise mathematical way to produce summations to any number of terms from the forward difference table.

Let	n =	1	2	3	4	5	6
	f(n) =	a_1	a_2	a_3	a_4	a_5	a_6
			b_1	b_2	b_3	b_4	b_5
				c_1	c_2	c_3	c_4
					d_1	d_2	d_3
						e_1	e_2

The sum of the series to n terms is given by

$a_1 \times {}^{n}/_{1!} + b_1 \times {}^{n(n-1)}/_{2!} + c_1 \times {}^{n(n-1)(n-2)}/_{3!}$ etc. until terms become zero.

This expression will give, for example, the summation of n cubic terms.

Let	n	=	1	2	3	4	5	6
n^3	=		1	8	27	64	125	216
				7	19	37	61	91
					12	18	24	30
						6	6	6

So summation to n terms is given by $1n + {}^{7n(n-1)}/_{6} + {}^{12n(n-1)(n-2)}/_{6}$

which expands out to $\frac{1}{4}(n^4 + 2n^3 + n^2) = \{\frac{1}{2}n(n+1)\}^2$

Relationship between Formula and Summation

Take the sequence

n	0	1	2	3	4	5	6	7
p	-1	1	3	5	7	9	11	13

and sum terms from n = 1 (setting p = 0 at n = 0) to give

p	0	1	4	9	16	25	36	49

now the forward difference table is

1	3	5	7	9	11	13

So the summation of a series is "one level up" and the formula for next term and summation of terms must be similar..

Let	n =	0	1	2	3	4	5	6
	f(n) =	a_0	a_1	a_2	a_3	a_4	a_5	a_6
			b_0	b_1	b_2	b_3	b_4	b_5
				c_0	c_1	c_2	c_3	c_4
					d_0	d_1	d_2	d_3

so the n^{th} term is given by $a_0/0! + b_0 n/1! + c_0\, n(n-1)/2! + d_0\, n(n-1)(n-2)(n-3)/3!$ …

and summation is given by $a_1 n/1! + b_1\, n(n-1)/2! + c_1\, n(n-1)(n-2)(n-3)/3!$ …

that is you omit the zeroeth term. Essentially you are moving down the second diagonal for summations instead of the first diagonal for the formula alone.

If you map out any typical quadratic and then build up the higher cubic that represents the summation of the quadratic terms it is self evident why you are now working down the second diagonal.

For a fuller mathematical explanation of this read my investigation "Falling Factorials".

rg