sin 18° and the golden ratio

sin 18°

let $\varphi = 18°$

then $5\varphi = 90°$

$5\varphi = 90° - 3\varphi$

$\sin 2\varphi = \sin(90° - 3\varphi)$

$\sin 2\varphi = \cos 3\varphi$

$2\sin\varphi\cos\varphi = 4\cos^3\varphi - 3\cos\varphi$

$2\sin\varphi = 4\cos^2\varphi - 3$

$2\sin\varphi = 4(1 - \sin^2\varphi) - 3$

$4\sin^2\varphi + 2\sin\varphi - 1 = 0$

$2\sin^2\varphi + \sin\varphi - ½ = 0$

$\sin\varphi = -1 \pm \sqrt{\{1 - (4)(2)(-½)\}}$

which all simplifies down to

$\sin\varphi = ¼(\sqrt{5} - 1)$

so $2\sin\varphi$ = golden ratio

cos 18°

$\cos^2\varphi = 1 - \sin^2\varphi$

$\cos\varphi = \sqrt{(1 - \sin^2\varphi)}$

$= \sqrt{\{1 - {}^1/_{16}(6 - 2\sqrt{5})\}}$

$= \sqrt{{}^1/_{16}(10 + 2\sqrt{5})}$

$= \sqrt{{}^1/_8(5 + \sqrt{5})}$

sin 36°

$\sin 2\varphi = 2\sin\varphi\cos\varphi$

$= 2(¼(\sqrt{5} - 1))\{\sqrt{{}^1/_8(5 + \sqrt{5})}\}$

which if you bother to work out is

$= ¼\sqrt{(10 - 2\sqrt{5})}$

cos 36°(2φ)

The easiest expression to work with is

$\cos 2\varphi = 1 - 2\sin^2\varphi$

$= 1 - 2(¼)^2(\sqrt{5} - 1)^2$

which simplifies to $¼(1 + \sqrt{5})$

sin 72°

again $\sin 4\varphi = 2\sin 2\varphi\cos 2\varphi$

$= 2(¼)¼\sqrt{(10 - 2\sqrt{5})}\{¼(1 + \sqrt{5})\}$

which simplifies to $¼\sqrt{(10 + 2\sqrt{5})}$

cos 72°

try $\cos^2\varphi = 1 - \sin^2\varphi$

$= 1 - {}^1/_{16}(10 + 2\sqrt{5})$

so $\cos\varphi = ¼\sqrt{(6 - 2\sqrt{5})}$

$= ¼\sqrt{\{(\sqrt{5} - 1)^2\}}$

$= ¼(\sqrt{5} - 1)$

Golden Ratio

$\sin 36° / \sin 72° = ¼\sqrt{(10 - 2\sqrt{5})} / ¼\sqrt{(10 + 2\sqrt{5})}$

$= \sqrt{\{{}^{(120 - 40\sqrt{5})}/_{80}\}}$

$= \sqrt{\{{}^{(3 - \sqrt{5})}/_2\}}$

$= ½\sqrt{\{(6 - 2\sqrt{5})^2\}}$

$= ½(\sqrt{5} - 1)$ *(see Appendix)*

which is the golden ratio.

If you draw a pentagon and connect the vertices the golden ratio crops up repeatedly. And this is why.

♓ rg

Appendix

That $(\sqrt{5} - 1)^2 = 6 - 2\sqrt{5}$ is interesting and I would have missed it if I didn't happen to know where I was heading and needed to end up.

The sequence continues

$(\sqrt{5} - 2)^2 = 9 - 4\sqrt{5}$

$(\sqrt{5} - 3)^2 = 14 - 6\sqrt{5}$

$(\sqrt{5} - 4)^2 = 21 - 8\sqrt{5}$

but none of these occur in the previous calculations so nothing missed.

rg