Mr G's Little Book on

Hypercomplex Numbers

Under what conditions does sin (A + B) = sin A + sin B ?

Preface

Nearly every text book on complex numbers will open with the statement that the imaginary number $i = \sqrt{^-1}$*. It may then warn about algebraic limitations on roots with a fallacious proof that* $^-1 = {^+1}$*.*

i is neither a number nor in any sense imaginary. It is defined by $i^2 = {^-1}$*. A square root is strictly defined as the positive value but is usually preceded by the symbol ±.*

i is best viewed as an operator on 1, rotating the number by convention through 90º anticlockwise. This defines the "Gaussian" complex plane.

However other "complex" planes can be constructed which this paper explores.

Introduction

I was investigating why there are three double angle cosine formulae but only one sine and concluded that it was our choice that we equated sine with the imaginary y-axis and cosine with the real x-axis. This did however lead me to discover there are other complex planes ;

for $z = a + ib$ where $i^2 + 1 = 0$ is the Guassian complex plane with which we are most familiar;

$z = a + wb$ where $w^2 = 0$ is the parabolic complex plane and

$z = a + r\,b$ (1) where $r^2 - 1 = 0$ $r \neq \pm 1$ is the hyperbolic complex plane where z is called a hyperbolic or bi-complex number – for reasons that will become apparent.

Summary

Given the ordered pair (a, b) where a and b both real and $r^2 - 1 = 0$ $r \neq \pm 1$ we have the hyperbolic number

$$z = a + r\,b.$$

There are practical uses for these other complex numbers. The Guassian complex plane is suited to mathematics of quantum mechanics but the hyperbolic complex plane is suited to the mathematics of special relativity – because there is a sign change in the way we treat time in the Lorentz transformation

$ds^2 = dx^2 + dy^2 + dz^2 - (ct)^2$ (2).

Four dimensional Minkowski spacetime is not the same as four dimensional Euclidean space.

In the investigation I derived the relationship

$\text{Sin}\,(u\,e_1 + v\,e_2) = \text{Sin}\,(u\,e_1) + \text{Sin}\,(v\,e_2)$

Investigation

First I established

$r = r \quad r^2 = 1 \quad r^3 = r$

$r^4 = 1 \quad r^5 = r$ etc.

$(-r) = -r \quad (-r)^2 = 1 \quad (-r)^3 = -r$

$(-r)^4 = 1 \quad (-r)^5 = -r$ etc.

The following relationships can now be confirmed :-

Conjugate

Define the conjugate $z^* = a - r\,b$ as for Guassian complex numbers.

Then it is easily established

$z\,z^* = |z|^2$ and $|z| = \sqrt{(a^2 - b^2)}$

Addition

setting $z = a + r\,b$ and $w = c + r\,d$

$z + w = w + z = (a + c) + r(b + d)$

Multiplication

$z \times w = w \times z = (ac + bd) + r(bc + ad)$

Inverse

$(a + r\,b)^{-1} = (a + r\,b) \;/\; (a^2 - b^2)$

but with the additional condition to Guassian complex numbers that $a \neq b$

This condition defines the light cones in Minkowski space-time diagrams.

Division

$z / w =$

$\{(ac - bd) + (bc - ad)\} / (c^2 - d^2)$

Square Root

Let $z = \sqrt{(a + r\,b)} = x + ry$ then

$x = \pm\sqrt{\tfrac{1}{2}\{a \pm \sqrt{(a^2 - b^2)}\}}$

and y follows immediately.

Commutativity of the Conjugate

It can easily be demonstrated that

$(z + w)^* = z^* + w^*$

$(z \times w)^* = z^* \times w^*$

$(z / w)^* = z^* / w^*$

Commutativity of the Modulus

It is also easily demonstrated

$|z \times w| = |z| \times |w|$

$|z / w| = |z| / |w|$

In Guassian complex numbers

$|z + w| = |z| + |w|$

if and only if $z = kw$ where k is real

$|z \times w|^a = |z|^a \times |w|^a$ but

$|z^a \times w^a| = |z|^a \times |w|^a$

has the restriction "**a**" real

and similar for hyperbolic numbers.

Taylor Series

Exponential

Let $e^z = 1 + z + z^2/2! + z^3/3! \ldots$

Assume $e^{rz} = 1 + rz + r^2z^2/2! + r^3z^3/3! \ldots$

$= 1 + r^2z^2/2! + r^4z^4/3! \ldots$

$+ \; r(z + z^3/3! + z^5/3! \ldots)$

so $e^{rz} = \cosh z + r \sinh z$

the hyperbolic equivalent of Euler's relationship.

Trigonometric

First I use capitals eg Sin specifically to indicate I am taking a complex value.

By a similar method to the exponential series it is easily established

Sin (r z) = r Sin (z)

Cos (r z) = Cos (z)

perhaps a surprising result

and for the hyperbolic functions

Sinh (r z) = r Sinh (z)

Cosh (r z) = Cosh (z)

Which means I can again derive

e^{rz} = Cosh z + r Sinh z

For trigonometric functions of the form Sin (a + r b) the best I could initially establish was

Sin (a+rb) = Sina Cosb + rCosa Sinb

Cos (a+rb) = Cosa Cosb – rSina Sinb

Sinh (a+r b) =

Sinh a Cosh b + r Cosh a Sinh b

Cosh (a + r b) =

Cosh a Cosh b + r Sinh a Sinh b

Idempotents

In ring theory (part of abstract algebra) an idempotent element, or simply an idempotent, of a ring is an element *a* such that $a^2 = a$.

let $(a + r b)^2 = a' + r b'$

so $a^2 + b^2 + 2abr = a' + r b'$

so $a^2 + b^2 = a'$

and $2ab = b'$

Hence a = ½ and b^2 = ¼ so our idempotents are

$e_1 = ½ (1 + r)$ and

$e_2 = ½ (1 – r)$

plus the trivials.

At this point note

$e_1 \times e_2 = 0$ and

$e_1 + e_2 = 1$

This is important later on. The first restricts hyperbolics to being a ring but not a field.

Trigonometric Alternate Forms

To obtain a more compact form I need to use the idempotents to transform

a + r b into the form

$a + r b = u e_1 + v e_2$

It is easily established

a = ½ (u + v)

b = ½ (u – v)

and therefore

u = a + b

v = a – b

Now I can see why hyperbolics can be termed split-complex because I am mapping a + r b onto two complex plains (a + b) and (a – b)

Trigonometric Functions

Now I can have another attempt at these functions. First I establish the relationships by writing out the Taylor series and extracting the common factor.

$\mathrm{Sin}\,(u\,e_1) = e_1\,\mathrm{Sin}\,u$

$\mathrm{Cos}\,(u\,e_1)$ is a little trickier because I don't have an e_1 in the first term so I have to subtract 1 and add back e_1 to produce

$\mathrm{Cos}\,(u\,e_1) = e_1\,(\mathrm{Cos}\,u - 1) + 1$

which looks ungainly but later is exactly what is required. Also note

$e_1\,\mathrm{Cos}\,u = \mathrm{Cos}\,(u\,e_1) + e_1 - 1$

By the same method I derive

$\mathrm{Sinh}\,(u\,e_1) = e_1\,\mathrm{Sinh}\,u$

and again the more ungainly

$\mathrm{Cosh}\,(u\,e_1) = e_1\,(\mathrm{Cosh}\,u - 1) + 1$

and hence

$e_1\,\mathrm{Cosh}\,u = \mathrm{Cosh}\,(u\,e_1) + e_1 - 1$

Now I'm ready to have another go at

$\mathrm{Sin}\,(a + r\,b) = \mathrm{Sin}\,(u\,e_1 + v\,e_2)$

$= \mathrm{Sin}\,(u\,e_1)\,\mathrm{Cos}\,(v\,e_2) +$

$\mathrm{Cos}\,(u\,e_1)\,\mathrm{Sin}\,(v\,e_2)$

$= e_1\mathrm{Sin}u\,\{e_2\,\mathrm{Cos}\,v - e_2 + 1\} +$

$\{e_1\,\mathrm{Cos}\,u - e_1 + 1\}\,e_2\,\mathrm{Sin}\,v$

Remembering $e_1 e_2 = 0$, I multiply out to $e_1\,\mathrm{Sin}\,u + e_2\,\mathrm{Sin}\,v$

Hence

$\mathrm{Sin}\,(ue_1 + ve_2) = \mathrm{Sin}\,(ue_1) + \mathrm{Sin}\,(ve_2)$

a neat and unexpected relationship. (3)

Similarly I establish

$\mathrm{Cos}\,(u\,e_1 + v\,e_2) = e_1\mathrm{Cos}\,u + e_2\,\mathrm{Cos}\,v$

The Cos functions are not quite so efficient at absorbing the idempotents and I get an annoying

$\mathrm{Cos}\,(u\,e_1 + v\,e_2) =$

$\mathrm{Cos}\,(u\,e_1) + \mathrm{Cos}\,(v\,e_2) + 1$

The associated hyperbolic functions of hyperbolic numbers are

$\mathrm{Sinh}\,(u\,e_1 + v\,e_2) =$

$\mathrm{Sinh}\,(u\,e_1) + \mathrm{Sinh}\,(v\,e_2)$

and

$\mathrm{Cosh}(u\,e_1 + v\,e_2) =$

$\mathrm{Cosh}(u\,e_1) + \mathrm{Cosh}\,(v\,e_2) - 1$

Note the minus sign

Conclusion and Further Study

I achieved a certain satisfaction in the Sin addition formula but was not able to relate the circular to hyperbolic functions as can be achieved in Gaussian complex numbers. I know hyperbolic numbers form a ring but not a field and need to study this further.

I need to understand the physical interpretation of

$\text{Sin}\,(u\,e_1 + v\,e_2) = \text{Sin}\,(u\,e_1) + \text{Sin}\,(v\,e_2)$

If I am to study their application in special relativity I am going to have to tackle differentiation of Guassian complex numbers first and revisit the Cauchy-Riemann equations and then carry that over to hyperbolic numbers.

References and Notes

[1] My acknowledgment to Mr. Philip Cutts for his suggestion to use the letter r

[2] http://en.wikipedia.org/wiki/Split-complex_number

[3] This equation is given at http://www.docin.com/p-154579804.html but not derived. The cosine formula omits the – 1 term. Despite extensive search I can find no other reference to these relationships.