Introduction to Propositional Calculus   Issue 1.1

Introduction

The works of two Greeks dominate Western Philosophy. We get Logic from Aristotle and Axiomatic Mathematics from Euclid. As Aristotle didn't like Maths (allegedly), Mathematics has been the poor relation until the start of the twentieth century when there was a move to bring Mathematics under Logic. In fact the paradoxes in language (such as "All Cretan are Liars" said by a Cretan) actually turned up in Mathematics when more powerful tools like Predicate Calculus were applied to arithmetic. Mathematicians were more than upset to discover their pet subject was either INCOMPLETE or INCONSISTENT. Fortunately none of that need concern the IB Studies student as far as exams are concerned. On the face of it, Propositional Calculus is reasonably straight-forward.

Symbols in Propositional Calculus

There are 6 symbols to learn.

Given the propositions **p** (eg "It is sunny") and **q** (eg "I ride my bike") then

$\neg p$	means	"It is not sunny".	**$\neg q$** means	"I don't ride my bike".
$p \vee q$	means	"Either it is sunny or I ride my bike <u>or both</u>"		Inclusive **OR**
$p \veebar q$	means	"Either it is sunny or I ride my bike <u>but not both</u>"		Exclusive **XOR**
$p \wedge q$	means	"It is sunny <u>and</u> I ride my bike"		**AND**
$p \Rightarrow q$	means	"p implies q".	But I can do whatever I like when it isn't sunny.	
$p \Leftrightarrow q$	means	"p and q are equivalent"	I <u>only</u> ride my bike when it's sunny.	

Truth Tables

Gottlob Frege developed Propositional Calculus purely in language. Ludwig Wittgenstein invented simple tables to achieve the same end. They follow from the axioms.

p	**q**	**$\neg p$**	**$p \vee q$**	**$p \veebar q$**	**$p \wedge q$**	**$p \Rightarrow q$**	**$p \Leftrightarrow q$**
T	T	F	T	F	T	T	T
T	F	F	T	T	F	F	F
F	T	T	T	T	F	T	F
F	F	T	F	F	F	T	T

Equivalences

p	**q**	**¬p**	**¬p v q**	**p ⇒ q**
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

1) Hence we observe that **¬ p v q** ≡ **p ⇒ q.** Hofstadter terms the "Switcheroo Rule".

p	**q**	**p $\underline{\vee}$ q**	**¬(p $\underline{\vee}$ q)**	**p ⇔ q**
T	T	F	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	T	T

2) Hence we observe that **¬ (p $\underline{\vee}$ q)** ≡ **p ⇔ q**

Some logicians define ⇒ and ⇔ in this way.

We could survive without these symbols but life's a lot easier if we include them.

p	**q**	**¬p**	**¬q**	**p ^ q**	**¬ (p ^ q)**	**¬p v ¬q**
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

3) Hence we observe that **¬ (p ^ q)** ≡ **¬p v ¬q**

p	**q**	**¬p**	**¬q**	**p v q**	**¬ (p v q)**	**¬p ^ ¬q**
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

4) Hence we observe that **¬ (p v q)** ≡ **¬p ^ ¬q**

These are de Morgan's rules. As the interpretation of the symbols v and **^** is arbitrary, once we've established the first equivalence, then the second must also be true because all we have to do is interchange v and **^.**

Axioms and Tautologies

Propositional Calculus is based on at least 3 axioms (unprovable truths). These are

Axiom of Identity **p = p**

Axiom of Non Contradiction **¬(p ^ ¬p)** ie **p** cannot be True and False at same time

Axiom of the Excluded Middle **p v ¬p** **ie** There's no third alternative.

The Law of Double Negation **¬ ¬ p = p** is a consequence of these axioms.

There are many other Laws that need not concern us at IB Studies level.

The "Non Contradiction" and "Excluded Middle" can be demonstrated by Truth Tables

p	**q**	**¬p**	**p ^ ¬p**	**¬ (p ^ ¬p)**	**p v ¬p**
T	T	F	F	T	T
T	F	F	F	T	T
F	T	T	F	T	T
F	F	T	F	T	T

When all four possibilities are TRUE, this is termed a tautology.

Truth Tables don't prove the Axioms. They "work" because we accept the axioms.

At IB studies level we needn't be too concerned which is the horse and which is the cart.

Alternative Logical Systems

Propositional Calculus offers a good baseline for the study of Logic but the Axioms are not universally accepted. The Axioms of Identity and Non Contradiction are challenged by the "Heap of Sand" proposition. If these "fall" then that brings **¬ ¬ p = p** into question.

Some mathematicians don't accept that **p v ¬p** is necessarily always true and therefore do not accept "proofs" by the "reductio ad absurdum" argument. Just because you might prove something isn't true doesn't necessarily mean it is true. It might be indeterminate.

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