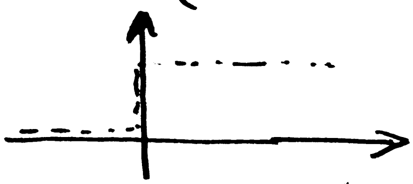
Laplace Transforms

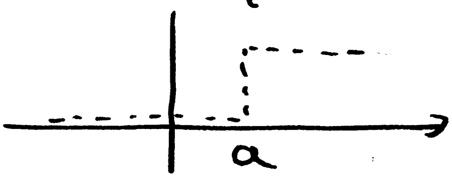
Unit Step function

$$u(t) = \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$$



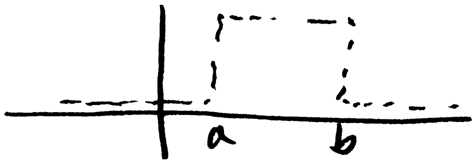
To shift the function to time $t=a$

$$u(t-a) = \begin{cases} 0 & t<a \\ 1 & t>a \end{cases}$$

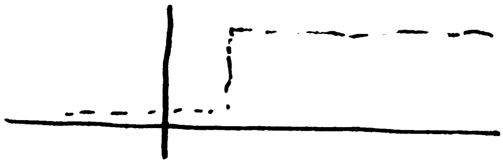


now to produce a step between a and b we say

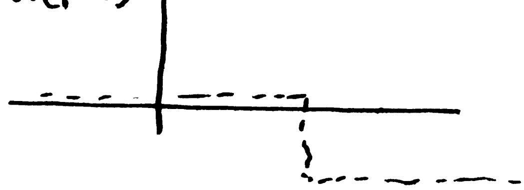
$$u(t-a) - u(t-b)$$



think of the two separate functions



$-u(t-b)$



so adding them together gives the step function between a and b for time $(b-a)$

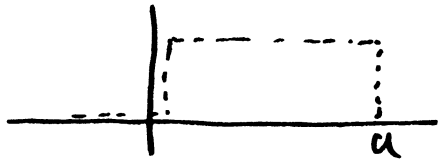
Ex

12 volts switched on at $t=4$

$$V(t) = 12 \times u(t-4)$$
$$= 12u(t-4)$$

Ex

Voltage 1 volt up to $t=a$ then switched off.



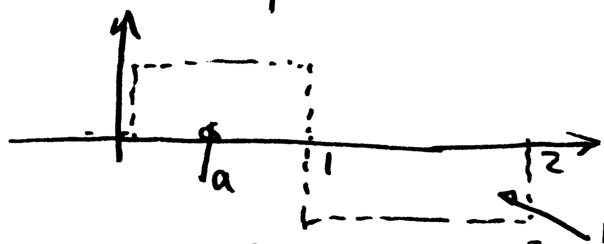
in general $V(t) = u(t-a) - u(t-b)$

in this case $V(t) = u(t) - u(t-a)$

Ex One cycle square wave

$f(0) = 4$ amplitude $= 4$

period $= 2$ sec.



a is $V_a(t) = 4\{u(t) - u(t-1)\}$

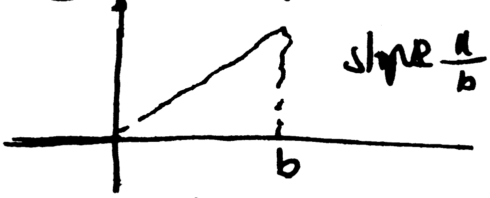
b is $V_b(t) = -4\{u(t-1) + u(t-2)\}$

$$V(t) = 4u(t) - 4u(t-1) - 4u(t-1) - 4u(t-2)$$
$$= 4u(t) - 8u(t-1) + 4u(t-2$$

Ex Ramp function $f(t) = t \quad t>0$

$$V(t) = tu(t)$$

Ex Sawtooth $f(t) = \frac{a}{b}t$



so a rectangular would be

$u(t) - u(t-b)$

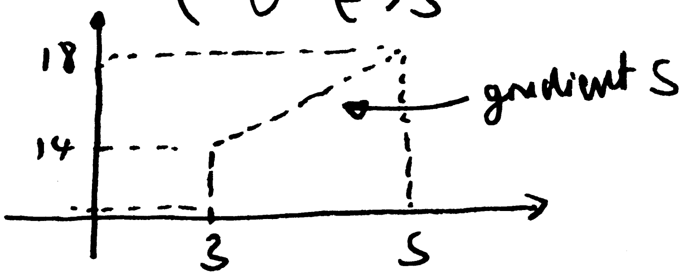
and we multiply by t for the ramp

$t\{u(t) - u(t-b)\}$

and by $\frac{a}{b}$ for the gradient

$V(t) = \frac{at}{b}\{u(t) - u(t-b)\}$

Ex $V(t) = \begin{cases} 0 & t < 3 \\ 2t+8 & 3 < t < 5 \\ 0 & t > 5 \end{cases}$



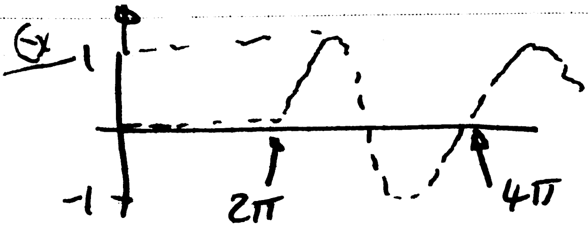
a step between 3 and 5 is

$u(t-3) - u(t-5)$

and we multiply that by the function $2t+8$

$V(t) = (2t+8)\{u(t-3) - u(t-5)\}$

Ex

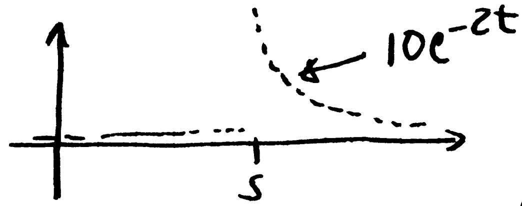


$f(t) = \sin t$

but we delay its start by the unit function $u(t-2\pi)$

so $V(t) = \sin t \cdot u(t-2\pi)$

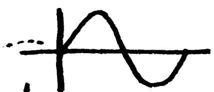
Ex



so $V(t) = 10e^{-2t}u(t-5)$

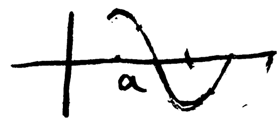
Distinguishing between

$f(t) \cdot u(t)$



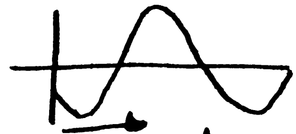
just starts at $t=0$

$f(t) \cdot u(t-a)$



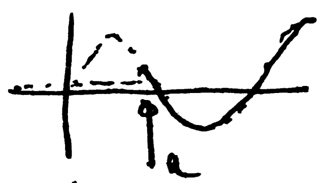
the function is delayed to time $t=a$

$f(t-a) \cdot u(t)$



the function itself is shift right by a but starts at $t=0$

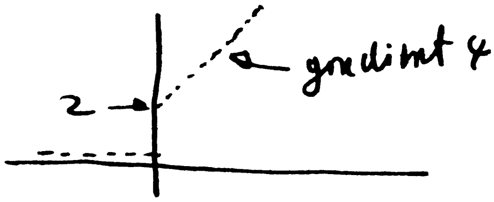
$f(t-a)\,u(t-a)$



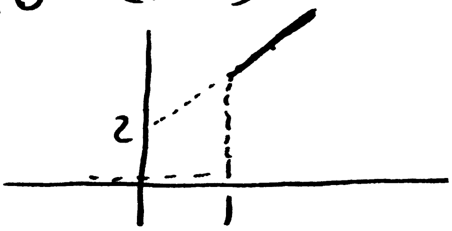
function delayed by a but also shifted.

Ex let $f(t) = 4t+2$
let $a = 1$

- $f(t)u(t) = (4t+2)u(t)$



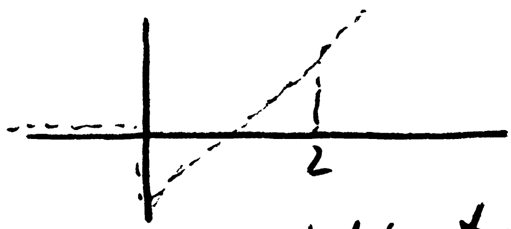
$V(t) = (4t+2)u(t-1)$



- function is delayed by 1 sec. but picks up at this point as if it had started at $t=0$

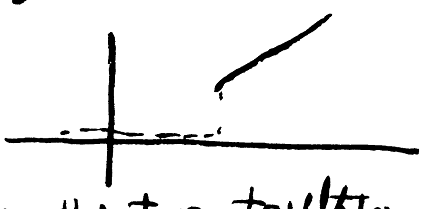
$V(t) = \{4(t-1)+2\}u(t)$

$= (4t-2)u(t)$

- 

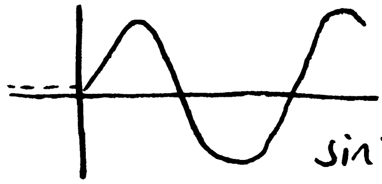
so we've shifted function along but it still switches on a $t=0$

$V(t) = (4t-2)u(t-1)$

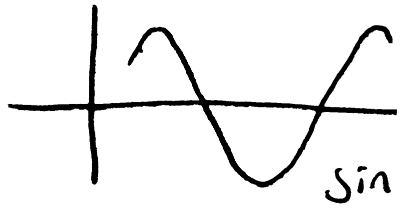


- so the two together shift the whole function along 1 and switches on at $t=1$.

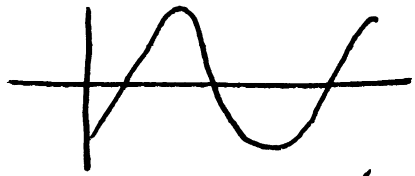
Ex $f(t) = \sin t$
$a = 0.7$



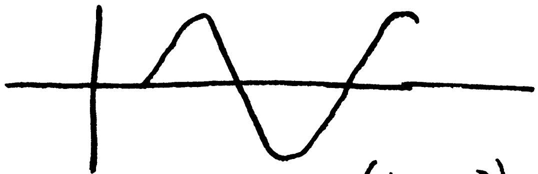
$\sin t\, u(t)$



$\sin t\, u(t-0.7)$



$\sin(t-0.7)u(t)$

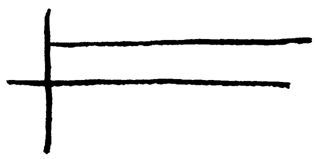


$\sin(t-0.7)u(t-0.7)$

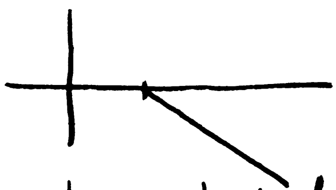
Ex

$f(t) = u(t) + (1-t)u(t-1)$
$+ (t-2)u(t-2)$

$u(t)$

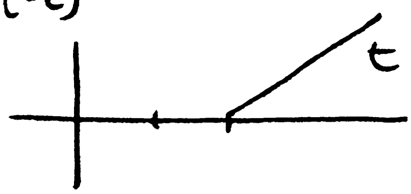


$u(t-1)(1-t)$

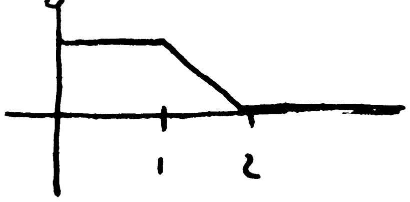


(remember it is virtual at $t=0$ and switches on a $t=1$)

$u(t-2)(t-2)$



so adding these three together we get



$$u(t) + (1-t)u(t-1) + (t-2)u(t-2)$$

we could try multiplying this all out

$v(t)$:

$$= u(t) + (1-t)u(t-1) + (t-2)u(t-2)$$

$$= u(t) + u(t-1) - tu(t-1) + tu(t-2) - 2u(t-2)$$

trick now is to let

$$u(t-1) = -u(t-1) + 2u(t-1)$$

so we can collect terms

$$[u(t)-u(t-1)] + 2[u(t-1)-u(t-2)] - t[u(t-1)-u(t-2)]$$

which is function we actually derived.

Laplace Transform

$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$\mathcal{L}\{f(t)\} = F(s)$

$\mathcal{L}\{g(t)\} = G(s)$

later we just say F or G.

unit step $f(t) = u(t)$

ramp $f(t) = t$

impulse $f(t) = \delta(t)$

Ex constant multiple

$\mathcal{L}\{7\sin t\} = 7\mathcal{L}\{\sin t\}$

Ex Linear Theorem

$\mathcal{L}(3t + 6t^2\}$

$= 3\mathcal{L}\{t\} + 6\mathcal{L}t^2$

Ex change scale

$\mathcal{L}f(6t) = \frac{1}{6}F(\frac{s}{6})$

Ex Shifting Theorem exponential damping

$\mathcal{L}\{e^{-3t}f(t)\} = F(s+3)$

Ex Differential Theorem.

$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s)$

Ex $f(t) = 4t^2$

$\mathcal{L}\{f(t)\} = 4\frac{2!}{s^3}$

$= \frac{8}{s^3}$

Ex $V(t) = 5\sin 4t$

$\mathcal{L}\{v(t)\} = 5\frac{4}{s^2+16}$

$= \frac{20}{s^2+16}$

Ex $g(t) = t\cos 7t$

$\mathcal{L}\{g(t)\} =$

$= \frac{s^2 - 7^2}{(s^2+7^2)^2}$

$= \frac{s^2 - 49}{(s^2+7^2)^2}$

or by the Differential Theorem

$\mathcal{L}\{\cos 7t\} = \frac{s}{s^2+7^2}$

then differentiate this (plus -)

$-\frac{(s^2+7^2) - s(2s)}{(s^2+7^2)^2}$

$= \frac{s^2 - 7^2}{(s^2+7^2)^2}$

~~Eg~~ $f(t) = e^{2t}\sin 3t$

from tables

$$\mathcal{L}\{f(t)\} = \frac{\cancel{\sin}\ w}{(s-2)^2+3^2}$$

but by the First Shifting Theorem let

$$\mathcal{L}\{g(t)\} = \frac{3}{s^2+9}$$

$(g(t) = \sin 3t)$

then we replace s with $s-w$

$$\mathcal{L}\{e^{2t}\sin 3t\} = \frac{3}{(s-2)^2+9}$$

Ex $f(t) = t^4 e^{-jt}$

let $g(t) = t^4$

$$\mathcal{L}\{g(t)\} = \frac{4!}{s^5}$$

now replace s with $s+j$

$$\mathcal{L}\{t^4(s-2)^2+9\} = \frac{24}{(s+j)^5}$$

Ex $f(t) = te^{-t}\cos 4t$

$g(t) = \cos 4t$

$$\mathcal{L}\{g(t)\} = \frac{s}{s^2+w^2}$$

$$\mathcal{L}\{e^{-t}\cos 4t\} = \frac{(s+1)}{(s+1)^2+4^2}$$

then we differentiate.

$$\frac{s^2-15}{(s^2+2s+17)^2}$$

Ex $f(t) = t^2\sin 5t$

$$\mathcal{L}\{t^2\sin 5t\}$$

$$= (-1)^2\frac{d^2}{ds^2}F(s)$$

$$\mathcal{L}\{\sin 5t\} = \frac{5}{s^2+25}$$

now differentiate twice.

$$\frac{(s^2+25)(0)-5(2s)}{(s^2+25)^2}$$

$$= \frac{-10s}{(s^2+25)^2}$$

and again.

$$\frac{(s^2+25)^2(-5)+10s(2(s^2+25))2s}{(s^2+25)^{2\,2}}$$

$$= \frac{30s^2-250}{(s^2+25)^3}$$

so $\mathcal{L}\{t^2\sin 5t\}$

$$= 10\,\frac{3s^2-25}{(s^2+25)^3}$$

Ex $t^3 \cos t$

to simplify

$\rightarrow t^2\, t \cos \cancel{3} t$

$\mathcal{L}\{t \cos \cancel{3} t\}$

$$\cancel{= \frac{s^2 - 9 1}{(s^2 + 9 1)^2}}$$

$$\frac{s^2-1}{(s^2+1)^2}$$

then differentiate twice to get

$$\frac{6s^4 - 36s^2 + 1}{(s^2+1)^4}$$

and $(-1)^2 = 1$

so $\mathcal{L}\{t^3 \cos t\}$

$$= \frac{6s^4 - 36s^2 + 1}{(s^2+1)^4}$$

Ex $\mathcal{L}\{\cos^2 3t\}$

now

$$\mathcal{L}\{\cos^2 t\} = \frac{s^2+2}{s(s^2+4)}$$

and Scale Property gives

$\mathcal{L}\{\cos^2 3t\} =$

$$\frac{1}{3}\; \frac{\left(\frac{s}{3}\right)^2 + 2}{\frac{s}{3}\left(\left(\frac{s}{3}\right)^2 + 4\right)}$$

$$= \frac{s^2+18}{s(s^2+36)}$$

Ex $\mathcal{L}\, u(t) = \frac{1}{s}$

$\mathcal{L}\, u(t-a) = e^{-as} \cancel{/s}$

Time Delay

$\mathcal{L}\, u(t-a)\, g(t-a)$

$= e^{-as}\, G(s)$

Ex

$$\mathcal{L}\, A[u(t-a) - u(t-b)]$$

$$= \frac{A[e^{-as} - e^{-bs}]}{s}$$

Ex

$$\mathcal{L}\{e^{t-a}(u(t-a) - u(t-b))\}$$

$$\left[\mathcal{L}\, e^t = \frac{1}{s-1}\right]$$

$= \mathcal{L}\{e^{t-a} u(t-a)\}$

$\cancel{- e^{b-a}}$

$- e^{b-a} \mathcal{L}\{e^{t-b} u(t-b)\}$

this is a little trick to get e^{t-b} inside bracket.

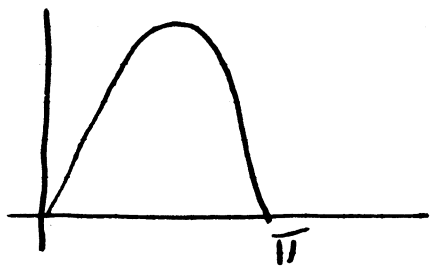
So now we use Time displacement

$$e^{-as} \frac{1}{s-1} - e^{b-a} e^{-bs} \frac{1}{s-1}$$

$$= \frac{e^{-as} - e^{b-a-bs}}{s-1}$$

(trust me it does)

Ex



$$f(t) = \sin t\,[u(t) - u(t-\pi)]$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\, u(t)\}$$

$$-\mathcal{L}\{\sin t\; u(t-\pi)$$

now here's a trick

$$\sin(t-\pi) = -\sin t$$

so we get

$$\mathcal{L}\{\sin t\, u(t)\}$$

$$+\mathcal{L}\{\sin(t-\pi)\, u(t-\pi)\}$$

$$= \frac{1}{s^2+1} + e^{-\pi s}\frac{1}{s^2+1}$$

$$= \frac{1+e^{-\pi s}}{s^2+1}$$

Solving DEs.

Ex

$$\frac{dy}{dt} + y = \sin 3t$$
$$y = 0 \quad t = 0$$

$$(sY - y(0)) + Y = \frac{3}{s^2+9}$$

$$(s+1)Y = \frac{3}{s^2+9}$$

$$Y = \frac{3}{(s^2+9)(s+1)}$$

$$= \frac{A}{s+1} + \frac{Bs+C}{s^2+9}$$

$$3 = A(s^2+9) + (s+1)(Bs+C)$$

equate s^2

$0 = A + B$

equate s

$0 = C + B$

equate s^0

$3 = 9A + C$

$A = \frac{3}{10}$ $B = \frac{-3}{10}$ $C = \frac{3}{10}$

so

$$Y = \frac{3}{10}\left(\frac{1}{s+1} - \frac{s}{s^2+9} + \frac{1}{s^2+9}\right)$$

$$y = \frac{3}{10}e^{-t} - \frac{3}{10}\cos 3t + \frac{1}{10}\sin 3t$$

Ex

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 0$$

at $y = 1$ $\frac{dy}{dt} = 0$ $t = 0$

$$s^2Y - sy(0) - y'(0) + 2(sY - y(0)) + 5Y = 0$$

$$(s^2Y - s) + 2(sY - 1) + 5Y = 0$$

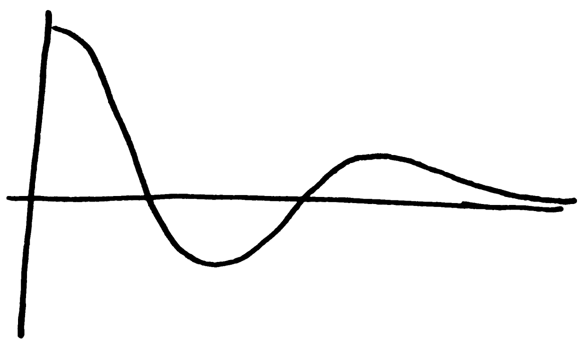
$$(s^2 + 2s + 5)Y = s + 2$$

$$Y = \frac{s+2}{s^2+2s+5}$$

$$= \frac{s+2}{(s+1)^2+4}$$

$$= \frac{s+1}{(s+1)^2+4} + \frac{1 \times 1 \times 2}{2(s+1)^2+4}$$

$$y = e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t$$



Ex

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = e^t$$

$$y = -2 \quad \frac{dy}{dt} = -3 \quad t = 0$$

$$\{s^2Y - sy(0) - y'(0)\}$$
$$-2(sY - y(0)) + Y$$
$$= \frac{1}{s-1}$$

$$(s^2Y + 2s + 3)$$
$$-2(sY + 2) + Y = \frac{1}{s-1}$$

$$(s^2 - 2s + 1)Y = \frac{1}{s-1} - 2s + 1$$

$$Y = \frac{\frac{1}{s-1} - 2s + 1}{s^2 - 2s + 1}$$

$$= \frac{\frac{1}{s-1} - 2s + 1}{(s-1)^2}$$

$$= \frac{1 - 2s(s-1) + (s-1)}{(s-1)^3}$$

$$= \frac{1 - 2s^2 + 2s + s - 1}{(s-1)^3}$$

$$= 1 - 2s^2 + 3s - 1$$

which the book rearranges to.

$$\frac{1}{2}\frac{2}{(s-1)^3} + \frac{-2s+1}{(s-1)^2}$$

Inverse transform

$$\frac{1}{2}e^t t^2$$

$$\frac{-2s+1}{(s-1)^2} = -\frac{2}{s-1} - \frac{1}{(s-1)^2}$$

$$\mathcal{L}^{-1} = -2e^t - te^t$$

$$y(t) = \frac{1}{2}t^2e^t - 2e^t - te^t$$

Ex

$$\frac{d^2i}{dt^2} + 2\frac{di}{dt} = \begin{Bmatrix} 0 & \text{at } t \leq 10 \\ 1 & 10 < t < 20 \\ 0 & t > 20 \end{Bmatrix}$$

$i(0) = 0 \quad i'(0) = 0$

$$\frac{d^2i}{dt^2} + 2\frac{di}{dt} = u(t-10) - u(t-20)$$

$$(s^2I - si(0) - i'(0)) + 2(sI - i(0)) = \frac{e^{-10s}}{s} - \frac{e^{-20s}}{s}$$

$$s^2I + 2sI = \frac{1}{s}\left(e^{-10s} - e^{-20s}\right)$$

$$I = \frac{e^{-10s} - e^{-20s}}{s^2(s+2)}$$

Now concentrate on $\frac{1}{s^2(s+2)}$

$$\frac{1}{s^2(s+2)} = -\frac{1}{4s} + \frac{1}{2s^2} + \frac{1}{4(s+2)}$$

$$\mathcal{L}^{-1} = -\frac{1}{4} + \frac{1}{2}t + \frac{1}{4}e^{-2t}$$

$$i(t) = -\frac{1}{4}u(t-10) + \frac{1}{2}(t-10)u(t-10) + \frac{1}{4}e^{-2(t-10)}u(t-10) - \left[-\frac{1}{4}u(t-20) + \frac{1}{2}(t-20)u(t-20) + \frac{1}{4}e^{-2(t-20)}u(t-20)\right]$$

not sure about this?

Ex

$$\frac{di}{dt} + 2i + 5\int_0^t i\,dt = u(t)$$

$$i(0) = 0$$

$$sI - i(0) + 2I + 5\frac{I}{s} = \frac{1}{s}$$

$$s^2 I + 2sI + 5I = 1$$

$$I = \frac{1}{s^2+2s+5}$$

$$= \frac{1}{(s+1)^2+4}$$

$$= \frac{1}{2}\,\frac{2}{(s+1)^2+2^2}$$

$$i = \frac{1}{2}e^{-t}\sin 2t$$