Further Differentiation

$\frac{d}{dx} \ln_e x$

Our only axiom is $\frac{d}{dx} e^x = e^x$ which is easily established from say the Taylor series for e^x

Let $x = e^y$

define $\ln x = \ln e^y$

$\ln x = y \ln_e e$

so $y = \ln x$

Now as $x = e^y$

then $\frac{dx}{dy} = e^y$

so $\frac{dy}{dx} = 1 / e^y$

so $\frac{dy}{dx} = \frac{1}{x}$

so we establish

$\mathbf{\frac{dy}{dx} \ln x = 1/x}$

$\frac{d}{dx} x^n$

The task is to derive this without using the power rule.

Let $y = x^n$

so $\ln y = \ln x^n$

so $\ln y = n \ln x$

so $\frac{d}{dx} = n$

$\frac{d}{dx}(\ln x) = \frac{n}{x}$ ❶

But we could also differentiate using the chain rule.

$\frac{d}{dx} \ln y = \{ 1/x^n \} \frac{d}{dx} x^n$ ❷

because $y = x^n$ but we don't yet know what $\frac{d}{dx} x^n$ is

But now we just equate ❶ and ❷

$\frac{n}{x} = \{ 1/x^n \} \frac{d}{dx} x^n$

$\frac{d}{dx} x^n = x^n \times \frac{n}{x}$

Therefore $\mathbf{\frac{d}{dx} x^n = n x^{n-1}}$

So although the consistency of maths requires this result it is nevertheless surprising how this expression can be so easily derived when the usual method is tedious manipulation of the binomial theorem.

$\frac{d}{dx} k^x$ where k = constant

Let $y = k^x$ so $\ln y = x \ln k$

$\frac{1}{y} \frac{dy}{dx} = \ln k$

$\mathbf{\frac{dy}{dx} = k^x \ln k}$

Alternative method

$\ln y = x \ln k$ so $x = \frac{\ln y}{\ln k}$

so $\frac{dx}{dy} = \frac{1}{y \ln k}$

so $\frac{dy}{dx} = y \ln k$

$\mathbf{\frac{dy}{dx} = k^x \ln k}$ as before

$^{d}/_{dx}\ k^{-x}$

Let $y = k^{-x}$

$\ln y = -x \ln k$

$^{1}/_{y}\ ^{dy}/_{dx} = -\ln k$

$\mathbf{^{dy}/_{dx} = -k^{x} \ln k}$

$^{d}/_{dx}\ k^{\wedge(1/x)}$

Let $y = k^{(1/x)}$

Let $\ln y = (^{1}/_{x}) \ln k$

Therefore $^{1}/_{y}\ ^{dy}/_{dx} = -^{\ln k}/_{x^2}$

$\mathbf{^{dy}/_{dx} = -k^{(1/x)}\ ^{\ln k}/_{x^2}}$

which is not particularly neat

$^{d}/_{dx}\ k^{\wedge(-1/x)}$

Let $y = k^{(-1/x)}$

Let $\ln y = -(^{1}/_{x}) \ln k$

Therefore $^{1}/_{y}\ ^{dy}/_{dx} = \ ^{\ln k}/_{x^2}$

$\mathbf{^{dy}/_{dx} = k^{(-1/x)}\ ^{\ln k}/_{x^2}}$

which at least has symmetry

$^{d}/_{dx}\ x^{x}$

Let $y = k^{(-1/x)}$

Let $\ln y = \ln x^{x}$

$\ln y = x \ln x$

$^{1}/_{y}\ ^{dy}/_{dx} = x\,(^{1}/_{x}) + \ln x$

$^{dy}/_{dx} = y\ (1 + \ln x)$

$\mathbf{^{dy}/_{dx} = x^{x}\ (1 + \ln x)}$

$^{d}/_{dx}\ x^{-x}$

Let $\ln y = -x \ln x$

$^{1}/_{y}\ ^{dy}/_{dx} = -x\,(^{1}/_{x}) - \ln x$

$^{dy}/_{dx} = -(x^{-x})\ (1 + \ln x)$

$^{d}/_{dx}\ x^{1/x}$

$y = x^{1/x}$

$\ln y = (^{1}/_{x}) \ln x$

$^{1}/_{y}\ ^{dy}/_{dx} = (^{-1}/_{x^2}) \ln x + {}^{1}/_{x^2}$

$^{1}/_{y}\ ^{dy}/_{dx} = (^{1}/_{x^2})\ (1 - \ln x)$

$\mathbf{^{dy}/_{dx} = x^{1/x}\ (^{1}/_{x^2})\ (1 - \ln x)}$

$^{d}/_{dx}\ x^{-1/x}$

$y = x^{-1/x}$

$\ln y = (^{-1}/_{x}) \ln x$

$^{1}/_{y}\ ^{dy}/_{dx} = (^{1}/_{x^2}) \ln x - {}^{1}/_{x^2}$

$^{1}/_{y}\ ^{dy}/_{dx} = (^{1}/_{x^2})\ (\ln x - 1)$

$\mathbf{^{dy}/_{dx} = x^{-1/x}\ (^{1}/_{x^2})\ (\ln x - 1)}$

$^{d}/_{dx}\ (x^{x})^{x}$

$y = (x^{x})^{x}$

$\ln y = x^2 \ln x$

$(^{1}/_{y})\ ^{dy}/_{dx} = {}^{x^2}/_{x} + 2x \ln x$

$\mathbf{^{dy}/_{dx} = (x^{x})^{x}\ (x + 2x \ln x)}$

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