12 Rules of Logarithms

Preliminaries

if $a = b^n$ ①

Taking logs of both sides we have

$\log_b a = \log_b b^n$

Assuming the truth of the power rule

$\log_b a = n \log_b b$

that is we can pull out the exponent inside a log expression and place it at the front.

Also remember

$\log_b b = 1$ ②

Hence $\log_b a = n$ ③

which is the definition of a log from ①

Substituting n back into ① gives

$b^\wedge \log_b a = a$ ④

Where b=10 then $\log_{10} a$ is written log a

Where b = e then $\log_e a$ is written ln a

and we have the key expression

$\mathbf{e^{\ln a} = a}$ ⑤

This shows that the log function is the inverse of the exponential function. We can now backtrack and formally state the 12 log rules. The first 3 are assumed and given without derivation.

• Product Rule

Rule 1 $\log_b (xy) = \log_b x + \log_b y$

Example $\log (2\times3) = \log 2 + \log 3$

• Quotient Rule

Rule 2 $\log_b (^x/_y) = \log_b x - \log_b y$

Example 1 $\log (^6/_3) = \log 6 - \log 3$

Example 2

We can combine both rules into one expression.

$\log 6 + \log 5 - \log 2 = \log 15$

which is true for any base.

• Power Rule

Rule 3 $\log_b a^k = k \times \log_b a$

Example let $6 = 2^x$

$\log 6 = \log 2^x$

$\log 6 = x \log 2$

$x = {}^{\log 6} / {}_{\log 2}$

Thus the power rule gives us a way in to solve exponential equations.

• Inverse Rule

set k = ⁻1 in Rule 3

$\log_b a^{-1} = {}^{-}1 \times \log_b a$

Rule 4 $\log_b (^1/_a) = {}^{-}\log_b a$

By this rule it can be seen logs for a < 1 are negative, log (1) = 0 and there are no logs for a ≤ 0 in the real domain.

• Log Product Rule

From ① $b^n = a$

Take logs of both sides to base c

$\log_c b^n = \log_c a$

$n \log_c b = \log_c a$

by ④ we recall

$n = \log_b a$

so $\log_b a \times \log_c b = \log_c a$

Rearrange to give

Rule 5 $\log_c b \times \log_b a = \log_c a$

Example $\log_3 4 \times \log_4 5 = \log_3 5$

which is easily recalled

• Base Change Rule

Rearrange Rule 5 to give

$\log_b a = \log_c a / \log_c b$

set c =10 such that $\log_{10} a$ is just log a

Rule 6 $\log_b a = {}^{\log a} / {}_{\log b}$

Most calculators do not have the option to calculate directly logs in bases other than 10 and e so just use the base change rule.

Example $\log_3 5 = {}^{\log 5}/ {}_{\log 3}$

• Power Base Rule *(tricky)*

Used when the base of the log may be expressed as a power.

Let $\log_{b^a} c = n$

$\log_{b^a} c = n \log_{b^a} b^a$

$c = b^{an}$

Hence $an = \log_b c$

substituting back the expression for n

$a \log_{b^a} c = \log_b c$

This derivation differs from those commonly given on the internet in that it introduces no superfluous variable.

Rule 7 $\log_{b^a} c = {}^1/_a \log_b c$

Example 1 $\log_{16} 5 = \frac{1}{4} \log_2 5$

and is easily verified by change of base rule.

Hint : Take the index of the base and pull forward to a fraction. Because we are taking the same number to a lower base we need to reduce proportionately the original value.

Example 2 $\log_{16} x = \log_2 5$

$\log_{16} x = \frac{1}{4} \log_2 x$

$\log_{16} x = \log_2 x^{1/4}$

$x^{1/4} = 5$

$x = 5^4$

$x = 625$

and the equation is solved without resorting to any logs because one base was a power of the other.

• Power Base Inverse Rule

set $a = {}^-1$

Rule 8 $\log_{1/b} c = {}^- \log_b c$

We can use this rule to change fractional power bases to "whole number" bases.

Example $\log_{1/2} 5 = {}^- \log_2 5$

and is easily verified by change of base rule

• Proportionality Rule

from Rule 5 $\log_c b \times \log_b a = \log_c a$

$\log_b a = \log_c a / \log_c b$

However $\log_c b$ is independent of a

Hence $\log_b a = k \log_c a$

Rule 9 $\log_b a \propto \log_c a$

where constant proportionality is $1 / \log_c b$

• Reciprocal Rule

from Rule 5 $\log_c b \times \log_b a = \log_c a$

Set c = a

$\log_a b \times \log_b a = \log_a a$ (ie 1)

Rule 10 $\log_a b = 1 / \log_b a$

Example $\log_3 5 = 1 / \log_5 3$

and this is immediately verified by using the change of base rule.

• Combination Rule

$\log_b a + k = \log_b a + k \log_b b$

$\log_b a + k = \log_b a + \log_b b^k$

Rule 11 $\log_b a + k = \log_b (a \times b^k)$

This is used to "pull in" a constant at the end of an expression into the log to simplify further analysis

• Exponential Rule

$a = b^{\log_b a}$ ④

set b = e

$a = e^{\ln a}$ given in ⑤

Rule 12 $a^x = e^{x \ln a}$

Example $3^2 = e^{2 \ln 3}$

which is easily verified as value 9 by calculator.

This is a key derivation as we can now standardise powers of arbitrary numbers into powers of e. This is required when we investigate raising complex numbers to complex powers.

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