Pentatope Numbers

The third diagonal of Pascal's Triangle gives the number sequence

1	5	15	35	70	126

If we construct a forward difference table and extend back a remarkable pattern emerges.

Start with	1	5	15	35	70	126
		4	10	20	35	56
			6	10	15	21
				4	5	6
					1	1

Now extend back that 4 5 6 sequence to 0 1 2 3 4 5 6

0	0	0	0	1	5	15	35	70	126
0	0	0	1	4	10	20	35	56	
0	0	1	3	6	10	15	21		
0	1	2	3	4	5	6			

So we have a sequence of numbers that appears to be a quartic and gives as the first four terms the value 0.

Taking the first term as n = 0 and using standard techniques the generating equation can be shown to be $(x^4 - 6x^3 + 11x^2 - 6x) \div 24$. But consider that factorisation is possible. As the terms are 0 for the first four terms that suggests the factors must be
x, $(x-1)$, $(x-2)$ and $(x-3)$.

By examination, it can be seen that the term $\div 24$ must be retained and indeed the equation is $(x^4 - 6x^3 + 11x^2 - 6x) \div 24 = x(x-1)(x-2)(x-3) \div 24$.

Pentatope numbers are the number of intersections of lines connecting all vertices of an n-gon, with the additional proviso that you need to make the n-gon slightly irregular so that every two-line intersection is discreet. The number of possible vertices of an n-gon is easily shown to be ½ n (n–3)

So the consequential number of intersections multiplies this by the additional term (n–2)(n–3)/12. I have yet to deduce exactly why this is so, but it is probably connected with Euler's formula and a bit of combinatorics.