Periodicity Trig Functions

Introduction

e^x = cosh x + sinh x

That is cosh x and sinh x are respectively the even part and odd part of the exponential function.

Now x $\rightarrow$ ix and assume the replacements are valid when transferring to the complex plane.

e^{ix} = cosh ix + sinh ix

but we already have by Euler

e^{ix} = cos x + i sinh x

so cosh ix = cos x and sinh ix = i sin x

Now x $\rightarrow$ $^x/_i$ and assume the replacements are valid when transferring to the complex plane.

cosh x = cos $^x/_i$

= cos (–ix)

and given that cos is an even function and holds in the complex plane

cosh x = cos ix

Similarly

sinh x = i sin $^x/_i$

= i sin (–ix)

and given that sin is an odd function and holds in the complex plane

sinh x = –i sin ix

Hyperbolic Periodicity

sinh (x + ½iπ)

= sinh x cosh ½iπ + cosh x sinh ½iπ

cosh ½iπ = cos ½π = 0

sinh ½iπ = i sin½π = i

so sinh (x + ½iπ) = i cosh x

sinh (x + iπ)

= sinh x cosh iπ + cosh x sinh iπ

cosh iπ = cos π = –1

sinh iπ = i sinπ = 0

so sinh (x + iπ) = – sinh x

sinh (x + $^3/_2$iπ)

= sinh x cosh $^3/_2$iπ + cosh x sinh $^3/_2$iπ

cosh $^3/_2$iπ = cos $^3/_2$π = 0

sinh $^3/_2$iπ = i sin$^3/_2$π = –i

so sinh (x + $^3/$iπ) = – i cosh x

sinh (x +2iπ)

= sinh x cosh2iπ + cosh x sinh2iπ

cosh 2iπ = cos 2π = 1

sinh 2iπ = i sinπ = 0

so sinh (x + 2iπ) = sinh x

Now we investigate cosh

cosh (x + ½iπ)

= coshx x cosh ½iπ + sinh x sinh ½iπ

cosh ½iπ = cos ½π = 0

sinhx ½iπ = i sin½π = i

so coshx (x + ½iπ) = i sinh x

$\cosh(x + i\pi)$

$= \cosh x \times \cosh i\pi + \sinh x\ \sinh x\ i\pi$

$\cosh i\pi = \cos\pi = -1$

$\sinh i\pi = i \sin\pi = 0$

so $\cosh(x + i\pi) = -\cosh x$

$\cosh x\ (x + {}^{3}/_{2}i\pi)$

$= \cosh x \times \cosh {}^{3}/_{2}i\pi + \sinh x\ \sinh x\ {}^{3}/_{2}i\pi$

$\cosh {}^{3}/_{2}i\pi = \cos {}^{3}/_{2}\pi = 0$

$\sinh {}^{3}/_{2}i\pi = i \sin {}^{3}/_{2}\pi = -i$

so $\cosh(x + {}^{3}/i\pi) = -i \sinh x$

$\cosh(x + 2i\pi)$

$= \cosh x\ \sinh 2i\pi + \sinh x\ \sinh 2i\pi$

$\cosh 2i\pi = \cos 2\pi = 1$

$\sinh 2i\pi = i \sin\pi = 0$

so $\cosh(x + 2i\pi) = \cosh x$

So sinh and cosh are perfectly symmetrical in this repect. For each ½iπ advance in phase cosh → sinh and sinh → cosh and the factor i is introduced. So after 4 × ½iπ the cycle completes.

Circular Periodicity

$\sin(x + \tfrac{1}{2}\pi)$

$= \sin x \cos \tfrac{1}{2}\pi + \cos x \sin \tfrac{1}{2}\pi$

$\cos \tfrac{1}{2}\pi = 0 \qquad \sin \tfrac{1}{2}\pi = 1$

so $\sin(x + \tfrac{1}{2}\pi) = \cos x$

$\sin(x + \pi)$

$= \sin x \cos\pi + \cos x \sin\pi$

$\cos\pi = -1 \qquad \sin\pi = 0$

so $\sin(x + \pi) = -\sin x$

$\sin(x + {}^{3}/_{2}\pi)$

$= \sin x \cos {}^{3}/_{2}\pi + \cos x \sin {}^{3}/_{2}\pi$

$\cos {}^{3}/_{2}\pi = 0 \qquad \sin {}^{3}/_{2}i\pi = -1$

so $\sin(x + {}^{3}/_{2}\pi) = -\cos x$

$\sin(x + 2\pi)$

$= \sin x \cos 2\pi + \cos x \sin 2\pi$

$\cos 2\pi = 1 \qquad \sin 2\pi = 0$

so $\sin(x + 2\pi) = \sin x$

Now we investigate cos

$\cos(x + \tfrac{1}{2}\pi)$

$= \cos x \cos \tfrac{1}{2}\pi - \sin x \sin \tfrac{1}{2}\pi$

$\cos \tfrac{1}{2}\pi = 0 \qquad \sin x\ \tfrac{1}{2}\pi = 1$

so $\cos x\ (x + \tfrac{1}{2}\pi) = -\sin x$

$\cos(x + \pi)$

$= \cos x \cos\pi - \sin x \sin\pi$

$\cos\pi = -1 \qquad \sin\pi = 0$

so $\cos(x + \pi) = -\cos x$

$\cos x\ (x + {}^{3}/_{2}i\pi)$

$= \cos x \cos {}^{3}/_{2}\pi - \sin x\ \sin x\ {}^{3}/_{2}\pi$

$\cos {}^{3}/_{2}\pi = 0 \qquad \sin {}^{3}/_{2}\pi = -1$

so $\cos(x + {}^{3}/_{2}\pi) = \sin x$

$\cos(x + 2\pi)$

$= \cos x \cos 2\pi - \sin x \sin 2\pi$

$\cos 2\pi = 1 \qquad \sin 2\pi = 0$

so $\cos(x + 2\pi) = \cos x$

So sin and cos display a certain asymmetry. For each ½π advance in phase cos → –sin but sin → cos. So after 4 × ½π the cycle completes.

Conclusion

Both sin/cos and sinh/cosh have a 2π / 2iπ periodicity but for slightly different reasons

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