An Investigation into Power Series – Method 1

Definitions

Define

$R_n = 1 + 2 + 3 + 4 + \dots n$

$S_n = 1\times2 + 2\times3 + 3\times4 + 4\times5 + \dots n(n+1)$

$T_n = 1\times2\times3 + 2\times3\times4 + \dots n(n+1)(n+2)$

$U_n = 1\times2\times3\times4 + \dots n(n+1)(n+2)(n+3)$

Conjecture

$R_n = \frac{1}{2} n (n+1)$

$S_n = {}^1/_3\, n (n+1) (n+2)$

$T_n = {}^1/_4\, n (n+1) (n+2) (n+3)$

$U_n = {}^1/_5\, n (n+1) (n+2) (n+3) (n+5)$

Proof by Induction

First step

To show the formulae are correct for n = 1. This is readily demonstrated as the last term cancel the denominator of the fraction and all that is left is the first term

Second step

To demonstrate

$\frac{1}{2} n (n+1) + (n+1) = \frac{1}{2} n (n+1) (n+2)$

${}^1/_3\, n (n+1) (n+2) + (n+1)(n+2)$

$= {}^1/_3\, n (n+1) (n+2) (n+3)$

${}^1/_4\, n (n+1) (n+2) (n+3)$

$+ (n+1) (n+2) (n+3)$

$= {}^1/_4\, n (n+1) (n+2) (n+3) (n+4)$

${}^1/_5\, n (n+1) (n+2) (n+3)$

$+ (n+1) (n+2) (n+3) (n+4)$

$= {}^1/_5\, n (n+1) (n+2) (n+3) (n+4) (n+5)$

As a preliminary to check on the right path I entered the left and right hand side of each equation into a graphical calculator and checked the tables generated were identical in each case.

Algebraically take the kth term

${}^1/_k\, n (n+1) (n+2) \dots (n+k-1)$

$+ (n+1) (n+2) \dots(n+k-1)$

$= (n+1) (n+2) \dots (n+k-1) ({}^n/_k + {}^k/_k)$

$= {}^1/_k\, (n+1) (n+2) \dots (n+k-1) (n+k)$

which was fairly painless.

Therefore we proved the conjecture by induction.

Power Series

The task is to establish power series for

$1^k + 2^k + 3^k + 4^k \dots$

First is to establish the summation formulae as polynomials by multiplying out the brackets.

$R_n = \frac{1}{2} \{ n^2 + n \}$

$S_n = {}^1/_3 \{ n^3 + 3n^2 + 2n \}$

$T_n = {}^1/_4 \{ n^4 + 6n^3 + 11n^2 + 6n \}$

$U_n = {}^1/_5 \{ n^5 + 10n^4 + 35n^3 + 50n^2 + 24n \}$

Now recall that

$$S_n = \sum n(n+1)$$
$$= \sum n^2 + \sum n$$
$$\sum n^2 = S_n - \sum n$$
$$\sum n^2 = S_n - R_n$$

Similarly

$$T_n = \sum n(n+1)(n+2)$$
$$= \sum (n^3 + 3n^2 + 2n)$$
$$= \sum n^3 + \sum 3n^2 + \sum 2n$$
$$\sum n^3 = T_n - \sum 3n^2 - \sum 2n$$

from whence we can deduce

$$\sum n^3 = T_n - 3 (S_n - R_n) - 2 R_n$$
$$\sum n^3 = T_n - 3 S_n + 2 R_n$$

So each time having established a new power series we use result to establish a further power series.

Summarising we establish

$$\sum n^2 = S_n - R_n$$
$$\sum n^3 = T_n - 3S_n + R_n$$
$$\sum n^4 = U_n - 6T_n + 7S_n - R_n$$
$$\sum n^5 = V_n - 10U_n + 25T_n - 155S_n - R_n$$

As an exercise I entered R_n into Y1, S_n into Y2, T_n into Y3, U_n into Y4 as was able demonstrate each summation was correct.

General Solution

From here the way to a general solution can be traced but it will not be pursued here. However as an introduction we can draw a table of indices

n^2	1	(1)				
n^3	1	(3)	1			
n^4	1	(6)	7	(1)		
n^5	1	(10)	25	(15)	1	
n^6	1	(15)	65	(90)	315	(1)

Patterns Observed

n^k has k terms

The first term is always 1.

The second term is the $(n-1)^{th}$ Δ number

Terms start +ve and then alternate in sign.

It is now conjectured that the 3rd term is the 4-D pyramidal number but higher dimensional pyramidal numbers are not particularly common. The left handed diagonal numbers are not recognised.

Power Series as Factored Expressions

$$\sum n = \tfrac{1}{2} n (n+1)$$
$$\sum n^2 = \tfrac{1}{6} n (n+1) (2n+1)$$
$$\sum n^3 = \{\tfrac{1}{2} n (n+1) \}^2$$
$$\sum n^4 = \tfrac{1}{30} n (n+1) (2n+1) (3n^2 + 3n - 1)$$
$$\sum n^5 = \tfrac{1}{12} n^2 (n+1)^2 (2n+1) (2n^2 + 2n - 1)$$
$$\sum n^6 = \tfrac{1}{12} n(n+1)(2n+1)(3n^4+6n^3-3n+1)$$
$$\sum n^7 = \tfrac{1}{24} n^2(n+1)^2(3n^4+6n^3-n^2-4n+2)$$

and further powers are available for reference on the internet.

To be continued …?

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