Mr G's Little Book on

Propositional Calculus

Preface

I wrote this booklet for my daughter Abigail when she was going for an interview to join the police and there was an element on Logic to be examined.

It didn't turn out to be quite as detailed as this.

Propositional Calculus

Propositional calculus uses letters to represent statements and then joins and manipulates those letters to form new expressions, which retain their common-sense interpretation.

Statements

Statements are represented by capital letters.

A stands for (say)
"Tom committed the crime"

B stands for
"Tom went to the pub last Thursday"

~B stands for
"Tom <u>didn't</u> go to the pub last Thursday"

This is the negation.

Be clear that statements can either be true or false. They start out as statements pure and simple.

Compound Sentences

Two statements can be combined together by a function to form a compound sentence. The combination implies of itself that it is its own "truth" This is called the "The Law of Identity". Whether the sentence in the final analysis is true or false depends on the truth or otherwise of the individual statements and the rules of the connecting function.

AND (Conjunction)

Tom is married and has kids.

A AND B.

Both have to be true for the compound sentence to be true.

If he isn't married or doesn't have kids or both then the whole expression is false.

OR (Disjunction)

Tom is either a thief or a liar

A OR B

If either is true the compound sentence is true. However **OR** is always inclusive. Tom can be a thief and a liar and the compound sentence is still true. So be careful if offering a precocious child ice cream or cake because she may take both. In such cases PC uses a specific function **XOR.**

Initial Pitfalls

There are two key pitfalls in all this.

1) Tertium non datur

There is no third option.

Either Dick went to the pub last Thursday or he didn't.

In PC **B OR ~B** is always true.

For the logicians among you this applies even when **B** is indeterminable and thus is termed two-valued logic.

If the TV detective is to be believed, all murder mysteries seem to hinge on the detective discovering that the case is an example of three-valued logic

If Dick went to the pub he didn't commit the murder. Yes he did, he slipped out the toilet window, knocked the guy over the head and slipped back before anyone missed him.

Notwithstanding that, all the following examples assume two-valued logic.

2) *Misinterpreting the Implication Rule*
If Tom forged the letter he'll have ink on his fingers.
If **A** is "forging" and
B is "ink on fingers"
PC writes this as **A** $\rightarrow$ **B** if A then B
But **B** can be for other reasons. If it weren't we'd call this biconditional.
Tom's got ink on his fingers so he forged the letter.
No he didn't – he's got ink on his fingers because his pen leaks.
In PC certain expressions are interchangeable

A $\rightarrow$ **B** is interchangeable with

~A OR B

If Tom forged the letter he'll have ink on his fingers

is the same as

Either Tom didn't forge the letter or he has ink on his fingers or, most importantly, <u>both</u>.
Which is exactly the case here.
Tom didn't forge the letter but he's also got ink on his fingers.
PC can show the false logic of assuming Tom forged the letter just because he's got ink on his fingers.

Proofs
There are seven proofs in Propositional Calculus.
In all the following exchanges be assured the desk sergeant always speaks the truth. The rookie PC often gets it wrong. For convenience I've assumed everyone's male.

1) Proof by Hypothetical Syllogism

PC Rookie *If Tom was there so was Dick*

Desk Sergeant

If Dick was there so was Harry.

PC Rookie

Tom was there (assume this true)

Desk Sergeant

Then Harry was there. Bring him in.

(A → B) AND (B → C)

→(A → C)

This is called the Theory of Consequences

2) Proof by Disjunctive Syllogism 1

Desk Sergeant

Either Tom or Dick committed the robbery.

PC Rookie

Tom was banged up in the cells that night.

Desk Sergeant

Dick did it. Bring him in.

(A OR B) AND ~B → A

This termed the *modus tollendo ponens* or "affirms by denying"

3) Proof by Disjunctive Syllogism 2

Desk Sergeant

You are George can't have your tea break at the same time.

PC Rookie

George is on his tea break

Desk Sergeant

Pick up your truncheon, son

(~A OR ~B) AND ~B → A

This termed the *modus ponendo tollens* or "denies by affirming"

4) Proof By Detachment

Desk Sergeant

If a jemmy was used it was forced entry

PC Rookie *A jemmy was used*

Desk Sergeant *It was forced entry.*

(A → B) AND A → B

This is termed the *modus ponendo ponens* or "affirms by affirming"

ie affirming the antecedent

But watch out for

Desk Sergeant *It was forced entry.*

PC Rookie *So a jemmy was used.*

Desk Sergeant *No he just barged in*

5) Proof By Indirect Reasoning

Desk Sergeant

Whoever forged this letter will have ink on his fingers.

PC Rookie

Tom doesn't have ink on his fingers

Desk Sergeant Better release him then

(A → B) AND ~B → ~A

But as given above don't jump to the false conclusion that anyone with ink on his fingers must have forged the letter. This is known as the *modus tollendo tollens* or "denies by denying"
ie denying the consequent.

6) Proof By Contradiction

This is trickier to demonstrate by a simple exchange but try this

PC Rookie
We should respect people's beliefs
Desk Sergeant
Harry is a paedophile. Do you respect his beliefs?
PC Rookie *No of course not*

Desk Sergeant
So you should respect people's beliefs and not respect people's beliefs. Perhaps you need to re-examine your original premise me lad.

In PC the compound sentence is

$(A \rightarrow B)$ AND $(A \rightarrow \sim B) \rightarrow \sim A$

If we assume **A** is true and that leads to concluding both **B** and **~B** then we must go back and examine our original assumption about **A**.
This is known as the *reductio ad absurdum* or "reducing to the absurb".
It is a common procedure in mathematics to assume a truth and then show that it leads to a contradiction this establishing the alternative.

7) Proof By Cases

PC Rookie
Either Tom or Dick did it that's for certain.
Desk Sergeant
Tom always works with Harry
PC Rookie
and Dick always works with Harry
Desk Sergeant *Better bring Harry in.*

(A OR B) AND
$[(A \rightarrow C)$ AND $(B \rightarrow C)] \rightarrow C$

The Implicit Function

Holmes turned over the piece of paper and examined it with his magnifying glass.

"See here Watson where the pens strokes don't join – clearly this is a forgery"

"Brilliant, Holmes, but who's the culprit, the Duchess or the Gamekeeper?"

"I suspect the Duchess but the proof will be a small ink stain on her little finger – see the writing is smudged here".

If only it were so easy. Fortunately the Duchess confessed immediately avoiding the need for a trial where any judge would have thrown out the "evidence". But let's examine in more detail what's happening here.

A Truth Table for two statements **A** and **B** gives all the outcomes for truth or falsehood of each statement.

A	**B**	**OR**
T	T	T
F	T	T
T	F	T
F	F	F

So the **OR** function is true if either **A** or **B** is true and only false if both **A** and **B** are false

A	**B**	**(A → B)**
T	T	T
T	F	F
F	T	T
F	F	T

So **A** → **B** means If **A** then **B**.

But **B** is not dependent solely on **A**. The Implicit function is only false if it specifically contradicts the premise that **B** is true when **A** is false.

In all other situations the Function is true.

Now **A** *(forged the letter)* is the major premise and **B** *(has ink on fingers)* the minor premise. But in our rush for the truth, seeing ink on the fingers gets us putting the cart before the horse.

It's safer to try and reconstruct the Implicit function in terms of the **OR** function. The **OR** function gives us three truths for one falsehood just as the Implicit function does so it's just a question of judicious rearrangement.

Leaving the outcome sequence as it is (T F T T) we just need to reverse the **A** sequence from T T F F to F F T T.

So we set up **~A** (not **A**). So now it's

~A	**B**	**OR**
F	T	T
F	F	F
T	T	T
T	F	T

So saying

"If they forged the letter then they have ink on their fingers"

is the same as saying

"Either they didn't forge the letter or they have ink on their fingers or both."

It's the "or both" that determines ink on your fingers and therefore doesn't necessarily mean you're guilty.

Hofstadter calls this the Switcheroo Rule after R. R. Switcheroo the Albanian railroad engineer who did Logic on the siding. The correct term for this interchangeability is the much less prosaic Implication Rule.

To keep the same outcome sequence T F T T we can also switch around **A** and **B** around and negate them. This is termed the contrapositive.

~B	***~A***	→
F	F	T
T	F	F
F	T	T
T	T	T

So saying

"If they forged the letter then they have ink on their fingers"

is the same as saying

"If they don't have ink on their fingers then they didn't forge the letter."

but the latter is better at encapsulating all that the sentence really has to say. For once the double negative, usually to be avoided, is much less likely to be misinterpreted.

However it's less interesting in crime novels to deduce the innocent rather than nail the guilty. The correct term for this interchangeability is the Transposition Rule.

And that's about it for the Implicit function. *rg*