Reduction Formulae

On the attached sheets are handwritten derivations of many of the reduction formulae with some special cases.

Here's a summary

Circular Functions

If $I_n = \int \sin^n x \, dx$

then $I_n = (^{-1}/_{n}) \sin^{n-1}x \cos x + (^{n-1}/_{n}) I_{n-2}$

If $I_n = \int \cos^n x \, dx$ **then**

$I_n = (^{1}/_{n+2}) \cos^{n-1}x \sin x + (^{n(n+1)}/_{n+2}) I_{n-2}$

If $I_n = \int \tan^n x \, dx$ **then**

$I_n = (^{1}/_{n-1})\tan^{n-1}x - I_{n-2}$

Inverse Circular Functions

If $I_n = \int \text{cosec}^n x \, dx$ **then**

$I_n = (^{-1)}/_{n-1})\text{cosec}^{n-2}x \cot x + (^{n-2}/_{n-1})I_{n-2}$

If $I_n = \int \sec^n x \, dx$ **then**

$I_n = (^{1)}/_{n-1})\sec^{n-2}x \tan x + (^{n-2}/_{n-1})I_{n-2}$

If $I_n = \int \cot^n x \, dx$ **then**

$I_n = (^{-1}/_{n-1})\cot^{n-1} x - I_{n-2}$

Hyperbolic Functions

If $I_n = \int \sinh^n x \, dx$ **then**

$I_n = (^{1}/_{n})\sinh^{n-1} x \cosh x - (^{n-1}/_{n})I_{n-2}$

If $I_n = \int \cosh^n x \, dx$ **then**

$I_n = (^{1}/_{n})\cosh^{n-1} x \sinh x - (^{n-1}/_{n})I_{n-2}$

If $I_n = \int \tanh^n x \, dx$ **then**

$I_n = (^{-1}/_{n-1}) \tanh^{n-1} x + I_{n-2}$

Inverse Hyperbolic Functions

If $I_n = \int \text{cosech}^n x \, dx$ **then**

$I_n = (^{-1}/_{n-1}) \text{cosech}^{n-1} \coth x + (^{n-2}/_{n-1})I_{n-2}$

If $I_n = \int \text{sech}^n x \, dx$ **then**

$I_n = (^{1}/_{n-1}) \text{sech}^{n-2} x \tanh x + (^{n-2}/_{n-1})I_{n-2}$

If $I_n = \int \coth^n x \, dx$ **then**

$I_n = (^{-1}/_{n-1}) \coth^{n-1} + I_{n-2}$

Exponential Functions

If $I_n = \int x^n e^x \, dx$ **then**

$I_n = x^n e^x - n I_{n-1}$

If $I_n = \int (\ln x)^n \, dx$ **then**

$I_n = x (\ln x)^n - n I_{n-1}$

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$$\text{Let } I_n = \int \sin^n x \, dx$$

$$= \int \sin^{n-1}x \, . \sin x \, dx$$

$$= -\sin^{n-1}x . \cos x + \int \cos x \{(n-1)\sin^{n-2}x \cos x\} \, dx$$

$$= -\sin^{n-1}x . \cos x + (n-1)\int \sin^{n-2}x \, (1-\sin^2 x) \, dx$$

$$= -\sin^{n-1}x . \cos x + (n-1)\int (\sin^{n-2}x . - \sin^n x) \, dx$$

$$= -\sin^{n-1}x . \cos x + (n-1)(I_{n-2} - I_n)$$

$$= -\sin^{n-1}x . \cos x + (n-1) I_{n-2} - (n-1) I_n$$

$$\therefore \; n I_n = -\sin^{n-1}x . \cos x + (n-1) I_{n-2}$$

$$\therefore \; I_n = \frac{-1}{n}\sin^{n-1}x . \cos x + \frac{n-1}{n} I_{n-2}$$

set $n=1$ gives $\int \sin x \, dx = -\cos x$

set $n=2$ gives $\int \sin^2 x \, dx = \frac{1}{2}(x - \sin x . \cos x.)$

$$= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)$$

set $n=3$ gives $\int \sin^3 x \, dx = -\frac{1}{3}\sin^2 x . \cos x - \frac{2}{3}\cos x$

$$= \frac{1}{3}\cos^3 x - \cos x ,$$

Let $I_n = \int \tan^n x \, dx$

$= \int \tan^{n-2} x . \tan^2 x \, dx$

$= \int \tan^{n-2} x \,(\sec^2 x - 1)\, dx$

$= \int (\tan^{n-2} x . \sec^2 x - \tan^{n-2} x)\, dx$

$= \frac{\tan^{n-1} x}{n-1} - I_{n-2}$

set $n=1$ gives $\int \tan \frac{x}{dx}$ = an indetemminate function.

set $n=2$ gives $\int \tan^2 \frac{x}{dx} = \tan x - x$

set $n=3$ gives $\int \tan^3 \frac{x}{dx} = \frac{\tan^2 x}{2} + \log_e \cos x$

set $n=5$ gives $\int \tan^5 x \, dx = \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x - \log_e \cos x$

Let $I_n = \int \operatorname{cosec}^n x \, dx$

$= \int \operatorname{cosec}^{n-2} x . \operatorname{cosec}^2 x \, dx$

$= -\operatorname{cosec}^{n-2} x . \cot x + \int \cot x \, (n-2) \operatorname{cosec}^{n-3} x . -\operatorname{cosec} x . \cot x \, dx$

$= -\operatorname{cosec}^{n-2} x . \cot x - (n-2) \int \cot^2 x . \operatorname{cosec}^{n-2} dx$

$= -\operatorname{cosec}^{n-2} x . \cot x - (n-2) \int (\operatorname{cosec}^2 x - 1) \operatorname{cosec}^{n-2} x \, dx$

$= -\operatorname{cosec}^{n-2} x . \cot x - (n-2) \int (\operatorname{cosec}^n x - \operatorname{cosec}^{n-2} x) \, dx$

$= -\operatorname{cosec}^{n-2} x . \cot x - (n-2) I_n + (n-2) I_{n-2}$

$\therefore (n-1) I_n = -\operatorname{cosec}^{n-2} x . \cot x + (n-2) I_{n-2}$

$\therefore I_n = \frac{-1}{n-1} \operatorname{cosec}^{n-2} x . \cot x + \frac{n-2}{n-1} I_{n-2}$

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set $n=2$ gives $\int \operatorname{cosec}^2 x \, dx = \frac{-1}{1} \operatorname{cosec}^0 x . \cot x + \frac{0}{1} I_{n-2}$

$= -\cot x .$

Let $I_n = \int \sec^n x \, dx$

$= \int \sec^{n-2} x \cdot \sec^2 x \, dx$

$= \sec^{n-2} x . \tan x - \int \tan x (n-2) \sec^{n-3} x . \sec x . \tan x \, dx$

$= \sec^{n-2} x . \tan x - (n-2) \int \tan^2 x \sec^{n-2} x \, dx$

$= \sec^{n-2} x . \tan x - (n-2) \int (\sec^2 x - 1) \sec^{n-2} x \, dx$

$= \sec^{n-2} x . \tan x - (n-2) \int (\sec^n x - \sec^{n-2} x) dx$

$= \sec^{n-2} x . \tan x - (n-2) \int \sec^n x \, dx + (n-2) \int \sec^{n-2} x \, dx$

$= \sec^{n-2} x . \tan x - (n-2) I_n + (n-2) I_{n-2}$

$\therefore (n-1) I_n = \sec^{n-2} x . \tan x + (n-2) I_{n-2}$

$I_n = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} I_{n-2}$

$\therefore$

set $n=2$ gives $\int \sec^2 x \, dx = \frac{1}{1} \sec^0 x . \tan x + \frac{0}{1} I_{n-2}$

$= \tan x$

set $n=3$ gives $\int \sec^3 x \, dx = \frac{1}{2} \sec x . \tan x + \frac{1}{2} \int \sec x$

$= \frac{1}{2} \sec x . \tan x + \frac{1}{2} \log_e (\sec x + \tan x)$

set $n=4$ gives $\int \sec^4 x \, dx = \frac{1}{3} \sec^2 x \tan x + \frac{2}{3} \tan x$

$= \tan x + \frac{1}{3} \tan^3 x$

$$I_n = \int \cot^n x \, dx$$

$$= \int \cot^{n-2} x \cdot \cot^2 x \, dx$$

$$= \int \cot^{n-2} (\operatorname{cosec}^2 x - 1) \, dx$$

$$= \int \cot^{n-2} x \cdot \operatorname{cosec}^2 x \, dx - \int \cot^{n-2} x \, dx$$

$$= \frac{-\cot^{n-1} x}{n-1} - I_{n-2}$$

$$I_n = \int \sinh^n x \, dx$$

$$= \int \sinh^{n-1} x . \sinh x \, dx$$

$$= \sinh^{n-1} x . \cosh x - \int \cosh x \, (n-1) \sinh^{n-2} x . \cosh x \, dx$$

$$= \sinh^{n-1} x . \cosh x - (n-1) \int \sinh^{n-2} x . \cosh^2 x \, dx$$

$$= \sinh^{n-1} x . \cosh x - (n-1) \int \sinh^{n-2} x . (1 + \sinh^2 x) \, dx$$

$$= \sinh^{n-1} x . \cosh x - (n-1) \int (\sinh^{n-2} x + \sinh^n x) dx$$

$$= \sinh^{n-1} x . \cosh x - (n-1) I_{n-2} + (n-1) I_n$$

$$\therefore n I_n = \sinh^{n-1} x . \cosh x - (n-1) I_{n-2}$$

$$\therefore I_n = \frac{1}{n} \sinh^{n-1} x . \cosh x - \left(\frac{n-1}{n}\right) I_{n-2}$$

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\begin{aligned}
I_n &= \int \cosh^n x \, dx \\
&= \int \cosh^{n-1} x \,.\, \cosh x \, dx \\
&= \cosh^{n-1} x \,.\, \sinh x - \int \sinh x \,(n-1)\cosh^{n-2} x \,.\, \sinh x \, dx \\
&= \cosh^{n-1} x \,.\, \sinh x - (n-1)\int \cosh^{n-2} x \,(1+\cosh^2 x)\, dx \\
&= \cosh^{n-1} x \,.\, \sinh x - (n-1) I_{n-2} + (n-1) I_n \\
n I_n &= \cosh^{n-1} x \,.\, \sinh x - (n-1) I_{n-2} \\
\therefore I_n &= \frac{1}{n}\cosh^{n-1} x \,.\, \sinh x - \frac{(n-1)}{n} I_{n-2}
\end{aligned}
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$$I_n = \int \tanh^n x$$

$$= \int \tanh^{n-2} x . \tanh^2 x \, dx$$

$$= \int \tanh^{n-2} x \, (1 - \operatorname{sech}^2 x) \, dx$$

$$= \int (\tanh^{n-2} x - \tanh^{n-2} x \operatorname{sech}^2 x) \, dx$$

$$= \frac{-\tanh^{n-1}}{n-1} + I_{n-2}$$

$$I_n = \int \operatorname{cosech}^n x \, dx$$

$$= \int \operatorname{cosech}^{n-2} x \operatorname{cosech}^2 x \, dx$$

$$= -\operatorname{cosech}^{n-2} x . \coth x + \int \coth x \,(n-2) \operatorname{cosech}^{n-3} x \times -\operatorname{cosech} x . \coth x \, dx$$

$$= -\operatorname{cosech}^{n-2} x . \coth x - (n-2) \int \coth^2 x . \operatorname{cosech}^{n-2} x . \, dx$$

$$= -\operatorname{cosech}^{n-2} x . \coth x - (n-2) \int (\operatorname{cosech}^2 x + 1) \operatorname{cosech}^{n-2} x \, dx$$

$$= -\operatorname{cosech}^{n-2} x . \coth x - (n-2) \int (\operatorname{cosech}^n x + \operatorname{cosech}^{n-2} x) \, dx$$

$$= -\operatorname{cosech}^{n-2} x . \coth x - (n-2) I_n - (n-2) I_{n-2}$$

$$\therefore (n-1) I_n = -\operatorname{cosech}^{n-2} x . \coth x - (n-2) I_{n-2}$$

$$\therefore I_n = \frac{-1}{n-1} \operatorname{cosech}^{n-2} x . \coth x - \frac{(n-2)}{(n-1)} I_{n-2}$$

set $n=2$ gives $\int \operatorname{cosech}^2 x \, dx = \frac{-1}{1} \operatorname{cosech}^0 x . \coth x - \frac{0}{1} I_{n-2}$

$$= -\coth x$$

$$I_n = \int \operatorname{sech}^n x \, dx$$
$$= \int \operatorname{sech}^{n-2} x . \operatorname{sech}^2 x \, dx$$
$$= \operatorname{sech}^{n-2} x . \tanh x + \int \tanh x \,(n-2) \operatorname{sech}^{n-3} x . \operatorname{sech} x \tanh x \, dx$$
$$= \operatorname{sech}^{n-2} x . \tanh x + (n-2) \int \tanh^2 x . \operatorname{sech}^{n-2} x \, dx$$
$$= \operatorname{sech}^{n-2} x . \tanh x + (n-2) \int (1 - \operatorname{sech}^2 x) \operatorname{sech}^{n-2} x \, dx$$
$$= \operatorname{sech}^{n-2} x . \tanh x + (n-2) \int (\operatorname{sech}^{n-2} x - \operatorname{sech}^n x) \, dx$$
$$= \operatorname{sech}^{n-2} x . \tanh x + (n-2) \int \operatorname{sech}^{n-2} x \, dx - (n-2) \int \operatorname{sech}^n x \, dx$$
$$(n-1) I_n = \operatorname{sech}^{n-2} x . \tanh x + (n-2) I_{n-2}$$
$$\therefore I_n = \frac{1}{n-1} \operatorname{sech}^{n-2} x . \tanh x + \frac{n-2}{n-1} I_{n-2}$$

set $n=2$ gives $\int \operatorname{sech}^2 x \, dx = \frac{1}{1} \operatorname{sech}^0 x . \tanh x + \frac{0}{1} I_{n-2}$

$$= \tanh x$$

$$I_n = \int \coth^n x \, dx$$

$$= \int \coth^{n-2} x \cdot \coth^2 x \, dx$$

$$= \int \coth^{n-2} x \cdot (1 + \operatorname{cosech}^2 x) \, dx$$

$$= \int \left(\coth^{n-2} x + \coth^{n-2} x \cdot \operatorname{cosech}^2 x\right) dx$$

$$= -\frac{\coth^{n-1} x}{n-1} + I_{n-2}$$

$$I_n = \int x^n e^x \, dx$$

$$= x^n e^x - \int e^x n x^{n-1} \, dx$$

$$= x^n e^x - n \int e^x x^{n-1} \, dx$$

$$= x^n e^x - n I_{n-1}$$

$$I_n = \int (\log x)^n \, dx$$

$$= \int (\log x)^{n-1} \log x \, dx$$

$$= (\log x)^{n-1}(x\log x - x) - \int (x\log x - x)(n-1)(\log x)^{n-2}\frac{1}{x}\, dx$$

$$= x(\log x)^n - x(\log x)^{n-1} - (n-1)\int (\log x - \cancel{\tfrac{1}{x}})(\log x)^{n-2}\, dx$$

$$= x(\log x)^n - x(\log x)^{n-1} - (n-1)\int \left[(\log x)^{n-1} - (\log x)^{n-2}\right] dx$$

$$= x(\log x)^n - x(\log x)^{n-1} - (n-1)\int (\log x)^{n-1}\, dx + (n-1)\int (\log x)^{n-2}\, dx$$

$$= x(\log x)^n - x(\log x)^{n-1} - (n-1) I_{n-1} + (n-1) I_{n-2}$$

$$= x(\log x)^n - n(I_{n-1}) - x(\log x)^{n-1} + I_{n-1} + (n-1) I_{n-2}$$

If we assume $I_n = x(\log x)^n - n I_{n-1}$

then last three terms $\rightarrow 0$

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