Mr G's Little Booklet on

Series and the Harmonic Triangle

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An Investigation into Series and the Harmonic Triangle

Summary

The author conducts a personal mathematical investigation every school holiday. Christmas 2012 investigation was to find the general solution for

$$\sum_{r=1}^{r=n} \frac{1}{r(r+1)(r+2)(r+3)\ldots(r+k)}$$

Initially I was unable to make substantial progress so as a diversion I investigated the Leibnitz's Harmonic triangle. In a truly serendipitous moment the solution suddenly dropped out of that second investigation.

$$\sum_{r=1}^{r=n} \frac{1}{r(r+1)(r+2)(r+3)\ldots(r+k)} = \frac{1}{k \times k!} - \frac{1}{k(n+1)(n+2)(n+3)\ldots(n+k)}$$

Introduction

I started with proofs of $\sum_{r=1}^{r=n} r = \frac{1}{2} n (n + 1)$

First Proof

Gauss's standard proof is

$\sum_{r=1}^{r=n} r = 1 + 2 + 3 + 4 + \ldots + n$

$\sum_{r=n}^{r=1} r = n + n-1 + n-2 + n-3 + \ldots + 1$

$2 \times \sum_{r=1}^{r=n} r = n (n + 1)$

$\sum_{r=1}^{r=n} r = \frac{1}{2} n (n + 1)$

Second Proof (Method of Differences or Telescoping)

Now $r^2 - (r - 1)^2 = 2r - 1$

So I set up a sequence of relationships for r = 1 to n

$1^2 - 0^2 = 2(1) - 1$

$2^2 - 1^2 = 2(2) - 1$

$3^2 - 2^2 = 2(3) - 1\ldots$

$n^2 - (n - 1)^2 = 2n - 1$

and summing left and right hand sides gives

$n^2 = 2 \sum n - n$

so by rearranging $\sum_{r=1}^{r=n} r = \frac{1}{2} n (n + 1)$

Third Proof (also by the Method of Differences)

This result gives me a clue to experiment with

$r (r + 1) - (r - 1) r = 2r$

Again I set up a sequence of relationships for r = 1 to n

$(1 \times 2) - (0 \times 1) = 2 \times 1$

$(2 \times 3) - (1 \times 2) = 2 \times 2$

$(3 \times 4) - (2 \times 3) = 2 \times 3\ldots$

$n (n + 1) - (n - 1) n = 2n$

summing left and right hand sides gives

$n (n + 1) = 2 \times \sum n$

so again I deduce $\sum_{r=1}^{r=n} r = \frac{1}{2} n (n + 1)$ (and see Appendix A)

General Solution $\Sigma\ r\ (r+1)\ (r+2)\ \ldots\ (r+k)$

I summarised my first theorem with two further known theorems.

$_{r=1}\Sigma^{r=n}\ r = \frac{1}{2}\ n\ (n+1)$

$_{r=1}\Sigma^{r=n}\ r\ (r+1) = \frac{1}{3}\ n\ (n+1)\ (n+2)$

$_{r=1}\Sigma^{r=n}\ (r+1)\ (r+2) = \frac{1}{4}\ n\ (n+1)\ (n+2)\ (n+3)$

and thus conjectured the general solution

$\Sigma\ r\ (r+1)\ (r+2)\ \ldots\ (r+k) = \frac{1}{(k+2)}\ n\ (n+1)\ (n+2)\ \ldots\ (n+k+1)$

General Solution $\Sigma\ r\ (r+k)$ (hereon for Σ assume $_{r=1}\Sigma^{r=n}$)

I summarised two known theorems

$\Sigma\ r\ (r+1) = \frac{1}{3}\ n\ (n+1)\ (n+2)$

$\Sigma\ r\ (r+2) = \frac{1}{6}\ n\ (n+1)\ (2n+7)$

The second I also derived by the Method of Differences by setting up the relationship

$r\ (r+1)\ (r+2) - (r-2)\ (r) + (r+2) = 6\ r\ (r+2)$ and proceeding as before but the algebra is detailed and it is not a particularly intuitive experience.

I then conjectured the general solution

$\Sigma\ r\ (r+k) = \frac{1}{6}\ n\ (n+1)\ (2n+3k+1)$

Setting $k = 0$ gave $\Sigma\ r^2 = \frac{1}{6}\ n\ (n+1)\ (2n+1)$ a known result

Setting $k = 1$ gave $\Sigma\ r\ (r+1) = \frac{1}{6}\ n\ (n+1)\ (2n+4)$

$= \frac{1}{3}\ n\ (n+1)\ (n+2)$ which was derived above.

However having established formulae for both $\Sigma\ r$ and $\Sigma\ r^2$ I can use this as a toolkit.

$\Sigma\ r\ (r+k) = \Sigma\ r^2 I + \Sigma\ r\ k$

$= \frac{1}{6}\ n\ (n+1)\ (2n+1) + k\ \frac{1}{2}\ n\ (n+1)$

$= \frac{1}{6}\ n\ (n+1)\ (2n+3k+1)$ as before.

General Solution $\Sigma\ {}^{1}/_{(r+k-1)(r+k)(r+k+1)}$

I summarised three known theorems

$\Sigma\ {}^{1}/_{r(r+1)} = {}^{n}/_{(n+1)}$ { *for proof see Appendix B* }

$\Sigma\ {}^{1}/_{(r+1)(r+2)} = {}^{n}/_{2(n+2)}$ { *actually* ${}^{(n+1)}/_{(n+2)} - {}^{1}/_{2}$ }

$\Sigma\ {}^{1}/_{(r+2)(r+3)} = {}^{n}/_{3(n+3)}$ { *actually* ${}^{(n+1)}/_{2(n+3)} - {}^{1}/_{6}$ }

I then conjectured the general solution

$\Sigma\ {}^{1}/_{(r+k)(r+k+1)} = {}^{n}/_{(k+1)(n+k+1)}$

General Solution $\Sigma\ {}^{1}/_{(r+k-1)(r+k)(r+k+1)}$

I summarised three known theorems

$\Sigma\ {}^{1}/_{r(r+1)(r+2)} = {}^{n(n+3)}/_{4(n+1)(n+2)}$

$\Sigma\ {}^{1}/_{(r+1)(r+2)(r+3)} = {}^{n(n+5)}/_{12(n+2)(n+3)}$

$\Sigma\ {}^{1}/_{(r+2)(r+3)(r+4)} = {}^{n(n+7)}/_{24(n+3)(n+4)}$

I then conjectured the general solution

$\Sigma\ {}^{1}/_{(r+k-1)(r+k)(r+k+1)} = {}^{n(n+2k+1)}/_{2n(n+1)(n+k)(n+k+1)}$

Central Investigation General Solution to $\Sigma\ {}^{1}/_{r(r+1)(r+2)\cdots(r+k)}$

I summarised my previously derived theorems

$$\Sigma\ {}^{1}/_{r(r+1)} = {}^{n}/_{(n+1)}$$

and $$\Sigma\ {}^{1}/_{r(r+1)(r+2)} = {}^{n(n+3)}/_{4(n+1)(n+2)}$$

and the general form would also need to demonstrate the harmonic series

$$\Sigma\ {}^{1}/_{r} = \infty$$

However no amount of judicious juggling would give me a workable general form. At this stage I had insufficient examples to guide me and had not fully developed the Method of Differences subsequently summarised in the appendices.

I subsequently discovered that the factoring of the numerator in the first two examples was purely fortuitous.

The Harmonic Triangle

As a distraction and in hope of later inspiration, I started to study Leibnitz's Harmonic Triangle. This is calculate by assuming the first diagonal proceeds $^1/_2$ $^1/_3$ $^1/_4$ $^1/_5$ etc. and each term is the sum of the two terms below – not above as in Pascal's triangle. So I started with

1			
$^1/_2$	$^1/_2$		
$^1/_3$	?	$^1/_3$	
$^1/_4$	?	?	$^1/_4$

and derived the triangle

1				
$^1/_2$	$^1/_2$			
$^1/_3$	$^1/_6$	$^1/_3$		
$^1/_4$	$^1/_{12}$	$^1/_{12}$	$^1/_4$	
$^1/_5$	$^1/_{20}$	$^1/_{30}$	$^1/_{20}$	$^1/_5$

As typing fractions was tedious I inverted the triangle

1				
2	2			
3	6	3		
4	12	12	4	
5	20	30	20	5

As typing out a triangle was also tedious I wrote each diagonal as follows

1st diagonal	1	2	3	4	5	…
2nd diagonal	2	6	12	20	30	…
3rd diagonal	3	12	30	60	105	…
4th diagonal	4	20	60	140	280	…
5th diagonal	5	30	105	280	630	…
6th diagonal	6	42	168	504	1260	…
7th diagonal	7	56	252	840	2310	…

Now I expected that the general term of each sequence would be an algebraic function, the 2^{nd} diagonal a quadratic, the 3^{rd} diagonal a cubic, the 4^{th} diagonal a quartic, the 5^{th} a quintic etc. and this proved to be correct.

Now as I had already developed an algorithm for determining the algebraic function for any finite sequence (*see Appendix C*), I just used Sloane's Encyclopaedia of Sequences, also thinking that if I came across an unregistered sequence I might be able to claim it as my own.

My results for each diagonal were

1^{st}	1	2	3	4	5…	n / 0!	Natural Numbers	A000027
2^{nd}	2	6	12	20	30…	n (n + 1) / 1!	Oblong Numbers	A002378
3^{rd}	3	12	30	60	105…	n (n + 1) (n + 2) / 2!		A002748
4^{th}	4	20	60	140	280…	n (n + 1) (n + 2) (n + 3) / 3!		A033488
5^{th}	5	30	105	280	630…	n (n +) (n + 2) (n + 3)(n + 4) / 4!		A174002
6^{th}	6	42	168	504	1260…	n(n+1)(n+2)(n+3)(n+4)(n+5) / 5!	unregistered	
7^{th}	7	56	252	840	2310…	n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)/6!	(ditto)	

For symmetry, I added division by 0! and 1! to the first two sequences.

I then summed the diagonals of the Harmonic triangle. I entered 20 rows into a spreadsheet and summed directly and discovered the summations were

Σ 1^{st} diagonal to infinity is infinity – this is the harmonic series.

Σ 2^{nd} diagonal to infinity is 1 Σ 3^{rd} diagonal to infinity is $^{1}/_{4}$

Σ 4^{th} diagonal to infinity is $^{1}/_{18}$ Σ 5^{th} diagonal to infinity is $^{1}/_{96}$

Next I noticed that the

2^{nd} diagonal was (1 × 2), (2 × 3), (3 × 4), (4 × 5), (5 × 6), …

3^{rd} diagonal was $^{(1\times2\times3)}/_{2!}$, $^{(2\times3\times4)}/_{2!}$, $^{(3\times4\times5)}/_{2!}$, $^{(4\times5\times6)}/_{2!}$, $^{(5\times6\times7)}/_{2!}$,…

4^{th} diagonal was $^{(1\times2\times3\times4)}/_{3!}$, $^{(2\times3\times4\times5)}/_{3!}$, $^{(3\times4\times5\times6)}/_{3!}$, $^{(4\times5\times6\times7)}/_{3!}$, ,… etc.

and that also gave the clue that the sequence 1, 4, 18, 96 …was $n \times n\,!$

Hence I could immediately see that the general formula for the summation

$\Sigma\ ^{1}/_{r\,(r+1)(r+2)\cdots(r+k)}$ was within my grasp. I finalised the detail in two steps.

Infinite Series

$^1/_{(1\times2)} + {}^1/_{(2\times3)} + {}^1/_{(3\times4)} \cdots \quad = (1 - {}^1/_2) + ({}^1/_2 - {}^1/_3) + ({}^1/_3 - {}^1/_4) \ldots$

$= 1$

$^1/_{(1\times2\times3)} + {}^1/_{(2\times3\times4)} + {}^1/_{(3\times4\times5)} \cdots = {}^1/_2({}^1/_{\{1\times2\}} - {}^1/_{\{2\times3\}}) + {}^1/_2({}^1/_{\{3\times4\}} - {}^1/_{\{4\times5\}})$

$\ldots$

$= {}^1/_4$

$^1/_{(1\times2\times3\times4)} + {}^1/_{(2\times3\times4\times5)} + {}^1/_{(3\times4\times5\times6)} \cdots \quad = {}^1/_3({}^1/_{\{1\times2\times3\}} - {}^1/_{\{2\times3\times4\}}) + {}^1/_3({}^1/_{\{2\times3\times4\}} -$

${}^1/_{\{3\times4\times5\}}) \ldots$

$= {}^1/_{18}$

and by the same method it can be demonstrated that

$^1/_{(1\times2\times3\times4\times5)} + {}^1/_{(2\times3\times4\times5\times6)} + {}^1/_{(3\times4\times5\times6\times7)} \cdots \quad = {}^1/_{96}$

and the general form is

$^1/_{(1\times2\times3\ldots\times k)} + {}^1/_{(2\times3\times4\times\ldots\times\{k+1\})} + {}^1/_{(3\times4\times5\times\ldots\times\{k+2\})} \cdots = {}^1/_{(n\times n!)}$

Finite Series

Now that I knew the infinite sums, I conjectured that the finite sums were in the form

eg. $\quad \Sigma\, {}^1/_{r(r+1)(r+2)} \quad = {}^{n(n+3)}/_{4(n+1)(n+2)}$

$\quad = {}^1/_4 - {}^1/_{f(n)}$

By algebraic manipulation I quickly established that

$\Sigma\, {}^1/_{r(r+1)} \quad = 1 - {}^1/_{(n+1)}$

$\Sigma\, {}^1/_{r(r+1)(r+2)} \quad = {}^1/_4 - {}^1/_{2(n+1)(n+2)}$

so I conjectured that

$\Sigma\, {}^1/_{r(r+1)(r+2)(r+3)} \quad = {}^1/_{18} - {}^1/_{3(n+1)(n+2)(n+3)}$

See *Appendix D* for proof by Method of Differences and *Appendix E* for proof by induction.

Conclusion General Solution $\sum_{r=1}^{r=n} \frac{1}{r(r+1)(r+2)(r+3)\ldots(r+k)}$

The algebraic manipulation enabled me to see that the general solution must be

$$\sum_{r=1}^{r=n} \frac{1}{r(r+1)(r+2)(r+3)\ldots(r+k)} = \frac{1}{k \times k!} - \frac{1}{k(n+1)(n+2)(n+3)\ldots(n+k)}$$

the desired result and achieved with no little satisfaction.

For k = 0 the harmonic series is indeed infinite.

Addendum 1

For completeness I then reduced the original unresolved case for k = 3 into a single fraction

$$\sum_{r=1}^{r=n} \frac{1}{r(r+1)(r+2)(r+3)} = \frac{1}{18} - \frac{1}{3(n+1)(n+2)(n+3)}$$

$$= \frac{3(n+1)(n+2)(n+3) - 18}{18 \times 3(n+1)(n+2)(n+3)}$$

$$= \frac{(n+1)(n+2)(n+3) - 6}{18(n+1)(n+2)(n+3)}$$

which reduces to $= \frac{(n^3 + 6n^2 + 11n)}{18(n+1)(n+2)(n+3)}$

and explains why I could not easily see the pattern from my original investigation, as it would seem the numerator factoring for k = 1,2 was merely fortuitous.

Addendum 2 General Term of the Harmonic Triangle

And finally having diverted to the Harmonic Triangle I should establish the general form B(n , r) that is n^{th} number in the r^{th} row.

Writing Pascals triangle and Leibnitz's (inverted) triangle side by side

1				1			
1	1			2	2		
1	2	1		3	6	3	
1	3	3	1	4	12	12	4

etc. and dividing Pascal's into Leibnitz's, I immediately derived

1			
2	2		
3	3	3	
4	4	4	4 etc.

so a bit of judicious algebraic manipulation gave me

$B (n , r) \times C (n , r) = {}^{1}/_{(n + 1)}$

Areas for Further Investigation

In a separate paper I investigated the extension of Pascal's triangle (the binomial distribution) into the Trinomial, Quadrinomial, Quintinomial and Sextinomial Triangle which gives the probability distribution for throwing many die. Could this be extended into the Trinomial Harmonic Triangle and beyond ?

Robert Goodhand

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Appendix A *Geometric "proof"* $\sum_{r=1}^{r=n} r = \frac{1}{2} n (n + 1)$

Start with the known fact that 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55

Imagine 55 squares stacked into a right angled isosceles triangle base 10 height 10.
As a first order approximation the area must be half base x height, 50 square units, which is the first term of the above expression.

The second term $^1/_2$ n, 5 units, is the missing 10 sliced diagonally squares that fall outside the triangle.

Alternatively I can fit together two of these triangles to make a rectangle 11 x 10 and the summation is immediately obvious. As a bonus it is now evident that the sum of any two triangular numbers is a square number.

The method can be extended to higher dimensions. A sequence of squares stacked into a pyramid will approximate to an area (a summation) of $^1/_3\ n^3$.

I conjectured many years ago that this pattern should continue to higher dimensions, so that in n dimensions the volume of a hyper pyramid is given by Hyper volume = $^1/_n$ x hyper base which I now understand is correct but not investigated.

In a separate investigation by the method of telescoping I deduced the formula for summing power series up to $1^{10} + 2^{10} + 3^{10} + \ldots n^{10}$ and discovered Bernouilli numbers. I was only mildly disappointed later to discover that Jacob Bernouilli had first identified these in 1737 because there was a mistake in his original publication.

I was once put in prefects' detention when about 15 and told to add up the numbers 1 to 100. I said "when I've done that can I go?" which agreed. So I first added up 1 to 9 and got 45. I then reasoned that pattern repeated 10 times so would be 450. I then realised in the "tens" column the total would be the same as it was the same numbers just in a different order – ten "1"s, ten "2"s etc. so again 450. Add to that the 45 gave me 4950 and finally the "100" to give 5050.

Appendix B To prove $\Sigma {}^{1}/_{r(r+1)} = {}^{n}/_{(n+1)}$

I set up $u_r = {}^{1}/_{r(r+1)}$

$u_{r+1} = {}^{1}/_{(r+1)(r+2)}$

so $u_r\, r(r+1) = u_{r+1}(r+1)(r+2)$

and therefore $r\, u_r = u_{r+1}(r+2)$

$r\, u_r = (r+1)\, u_{r+1} + u_{r+1}$

$r\, u_r - (r+1)\, u_{r+1} = u_{r+1}$

This term "$r\, u_r - (r+1)\, u_{r+1}$" is the key because I now set up a sequence from u_1 to u_{n-1}

$1\, u_1 - 2\, u_2 = u_2$

$2\, u_2 - 3\, u_3 = u_3$

$3\, u_3 - 4\, u_4 = u_4 \ldots$

$(n-1)\, u_{(n-1)} - n\, u_n = u_n$

I then added the left and right columns to get

$1\, u_1 - n\, u_n = (u_2 + u_3 + u_4 + u_5 \ldots u_n)$

Adding u_1 to both sides $2\, u_1 - n\, u_n = (u_1 + u_2 + u_3 + u_4 \ldots u_n)$

and thus $\Sigma u_r = {}^{2}/_{(1\times 2)} - {}^{n}/_{n(n+1)}$

hence $\Sigma u_r = {}^{n}/_{(n+1)}$ the desired result.

Appendix C Summing an Algebraic-Power Series

Given the terms of an algebraic function f (n)

n	1		2		3		4	…	n
f (n)	a_1		a_2		a_3		a_4	…	a_n
1st diff.			b_1		b_2		b_3		
2nd diff.				c_1		c_2			
3rd diff.					d_1				

Then ${}_{r=1}\Sigma^{r=n} a_r = a_1\,(n)/1! + b_1\,(n)(n-1)/2! + c_1\,(n)(n-1)(n-3)/3! \ldots$

This sequence will eventually terminate with 0 terms for any algebraic function.

Appendix D – Obtaining Series by the Method of Differences

I subsequently realised there was a direct way to sum $^1/_{r(r+1)(r+2)(r+3)}$ to n terms.

$$u_r = {}^1/_{r(r+1)(r+2)(r+3)}$$

$$u_{r+1} = {}^1/_{(r+1)(r+2)(r+3)(r+4)}$$

$$u_r\, r(r+1)(r+2)(r+3) = u_{r+1}(r+1)(r+2)(r+3)(r+4)$$

$$r\, u_r = u_{r+1}(r+4)$$

$$r\, u_r = (r+1)\, u_{r+1} + 3\, u_{r+1}$$

$$r\, u_r - (r+1)\, u_{r+1} = 3\, u_{r+1}$$

This term "$r\, u_r - (r+1)\, u_{r+1}$" is the key because I now set up a sequence from u_1 to u_{n-1}

$1\, u_1 - 2\, u_2 = 3\, u_2$

$2\, u_2 - 3\, u_3 = 3\, u_3$

$3\, u_3 - 4\, u_4 = 3\, u_4 \ldots$

$(n-1)\, u_{(n-1)} - n\, u_n = 3\, u_n$

So using essential the same technique I add the two columns to get

$1\, u_1 - n\, u_n = 3\,(u_2 + u_3 + u_4 + u_5 \ldots u_n)$

Adding $3\, u_1$ to both sides

$4\, u_1 - n\, u_n = 3\,(u_1 + u_2 + u_3 + u_4 \ldots u_n)$

and thus $\Sigma u_r = {}^1/_3\,(4\, u_1 - n\, u_n)$

$$\Sigma u_r = {}^1/_3\,({}^4/_{(1\times2\times3\times4)} - {}^n/_{n(n+1)(n+2)(n+3)}$$

$$\Sigma u_r = {}^1/_{18} - {}^1/_{3(n+1)(n+2)(n+3)}$$

the desired result.

This procedure is more robust in following a structured method for determining the difference term rather than just producing $u_r = f(r) - f(r+1)$ like a rabbit out of a hat.

In this case $f(r) = {}^1/_{3(r+1)(r+2)(r+3)}$.

Appendix E Proof by Induction $\Sigma\, {}^1/_{r(r+1)(r+2)(r+3)} = {}^1/_{18} - {}^1/_{3(n+1)(n+2)(n+3)}$

This is true for r = 1, n = 1 so for proof by induction assume

if $\quad {}_{r=1}\Sigma^{r=n}\ {}^1/_{r(r+1)(r+2)(r+3)} = {}^1/_{18} - {}^1/_{3(n+1)(n+2)(n+3)}$

then $\quad {}_{r=1}\Sigma^{r=n+1}\ {}^1/_{r(r+1)(r+2)(r+3)} = {}_{r=1}\Sigma^{r=k}\ {}^1/_{r(r+1)(r+2)(r+3)} + {}^1/_{(n+1)(n+2)(n+3)(n+4)}$

$$= {}^1/_{18} - {}^1/_{3(n+1)(n+2)(n+3)} + {}^1/_{(n+1)(n+2)(n+3)(n+4)}$$

$$= {}^1/_{18} - {}^1/_{(n+1)(n+2)(n+3)}\left\{ {}^1/_3 - {}^1/_{(n+4)} \right\}$$

$$= {}^1/_{18} - {}^1/_{(n+1)(n+2)(n+3)}\left\{ {}^{(n+4-3)}/_{3(n+4)} \right\}$$

$$= {}^1/_{18} - {}^{(n+1)}/_{3(n+1)(n+2)(n+3)(n+4)}$$

$$= {}^1/_{18} - {}^{(n+1)}/_{3(\{n+1\}+1)(\{n+1\}+2\}\{\{n+1\}+3)}$$

the desired result.

Appendix F Why does 0! =1

It is often stated that 0! = 1 by definition. This is misleading for 2 reasons.

Mathematically the Factorial function is part of the Gamma function for positive integers and 0! follows from this.

However it can be immediately derived by a shift in definition.

Let $\quad$ 3! = 4! / 4

so $\quad$ 2! = 3! / 3

and $\quad$ 1! = 2! / 2

therefore $\quad$ 0! = 1! / 1 = 1

Simple.