Solving the Quadratic

Introduction

Suppose we define a quadratic function $f(x)$ as $f(x) = ax^2 + bx + c$.

That's not really an equation in the usual sense. It's simply a definition of the function $f(x)$.

Now if we set $f(x) = 0$ the value of x that sets $f(x) = 0$ is called **the solution** of $f(x)$. So $ax^2 + bx + c = 0$ is now **an equation** and the value of x is the solution.

Rearranging we have $c = -ax^2 - bx$
We have made c – normally considered a constant – as the **dependent variable** of the equation. That is for different values of the variable x we adjust c to provide the solution.

Now by clever manipulation we can make x the subject of this equation and we have

$$x = \{-b \pm \sqrt{(b^2 - 4ac)}\} / 2a$$

The question is how exactly was this miracle of manipulation achieved?

The solution of the quadratic

Let $f(x) = ax^2 + bx + c$
divide through by a such that
$g(x) = x^2 + {}^{b}/_{a}\, x + {}^{c}/_{a}$

Now $g(x)$ is a different function to $f(x)$ but they happen to have the same solution. To see exactly what's happening experiment with
$Y1 = 2x^2 + 4x - 6$ and
$Y2 = x^2 + 2x - 3$ on your TI-83/84.

For the solution of $g(x)$ set
$x^2 + {}^{b}/_{a}\, x + {}^{c}/_{a} = 0$

but how to extract x when it's embedded in two terms at different powers?

Let's experiment with
$(x + {}^{b}/_{2a})^2 = x^2 + {}^{b}/_{a}\, x + {}^{b^2}/_{4a^2}$
and in one stroke we have an expression that matches the first two term of $g(x)$. So
$(x + {}^{b}/_{2a})^2 - {}^{b^2}/_{4a^2} + {}^{c}/_{a} = x^2 + {}^{b}/_{a}\, x + {}^{c}/_{a}$

Hence $g(x) = (x + {}^{b}/_{2a})^2 - {}^{b^2}/_{4a^2} + {}^{c}/_{a}$
and for the solution we set
$(x + {}^{b}/_{2a})^2 - {}^{b^2}/_{4a^2} + {}^{c}/_{a} = 0$
$(x + {}^{b}/_{2a})^2 \quad = {}^{b^2}/_{4a^2} - {}^{c}/_{a}$
$(x + {}^{b}/_{2a}) \quad = \sqrt{({}^{b^2}/_{4a^2} - {}^{4ac}/_{4a^2})}$
$x \quad = -{}^{b}/_{2a} \sqrt{({}^{b^2}/_{4a^2} - {}^{4ac}/_{4a^2})}$
and hence $x = \{-b \pm \sqrt{(b^2 - 4ac)}\} / 2a$

Homework

If $f(x) = ax^3 + bx^2 + cx + d$
find an expression for x such that $f(x) = 0$
Time allowed say 20 years which is how long it took Gerolama Cardano with help from Scipione del Ferro and Niccolo Tartaglia.

The solution of the cubic leads to a **"radical"** solution for the quartic
$g(x) = ax^4 + bx^3 + cx^2 + dx + e$
Time allowed say another 20 years.

The quintic is
$h(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$

To solve this in radicals you can have as much time as you like as there isn't one.

This was hinted at by Evariste Galois in scribbled notes he made the night before – inconveniently for mathematics – he died in a duel the next day aged just 21.
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