Mr G's Little Booklet on

The Axioms of Special Relativity

and why $e = mc^2$

Einstein's Special Theory of Relativity is based on one definition, one principle and two axioms.

Definition

A system (e.g. a laboratory, individual, or ensemble) is termed an inertial system or an inertial frame of reference if it may be viewed as either in a state of rest or in uniform motion.

Principle of the Invariance of Coincidences

When one observer says two events coincide in space and time, so will all other observers. (If two cars smash into each other, everyone who sees it will agree what happened.)

Axiom 1 A system moving uniformly with respect to an inertial system is also an inertial system.

Corollary 1.1 One cannot tell by experiment whether one is at rest or moving uniformly.

Corollary 1.2 If two experiments are performed under identical conditions except that one is done in a laboratory at rest and the other in a uniformly moving laboratory, the two experiments will lead to exactly the same conclusions.

Corollary 1.3 The laws of nature are the same in a laboratory at rest as they are in a uniformly moving laboratory.

Corollary 1.4 Anybody moving uniformly with respect to somebody at rest is entitled to consider himself to be at rest and the other person to be moving uniformly.

Axiom 2 Anything that moves past a given inertial observer with the speed c moves past any other inertial observer with the same speed c.

Corollary 2.1 The speed of light, c, has the same value with respect to any inertial observer.

From these two axioms the following five rules may be deduced.

Rule 1 A metre stick moving with uniform velocity in a direction perpendicular to itself has the same length as a metre stick at rest. (This is known as the principle of rotational invariance).

Rule 2 If a clock takes a time t between ticks when it is stationary, then when it moves with velocity v, it takes a longer time $t/\sqrt{(1-v^2/c^2)}$.
This is the time dilation effect

Rule 3 If a metre stick has a length k when it is stationary, then when it moves with uniform velocity v in a direction parallel to its length, it contracts to the length $k/\sqrt{(1-v^2/c^2)}$.
This is the Fitzgerald contraction.

Rule 4 If two clocks are synchronised and separated by a distance k in their proper frame, then in an inertial frame in which the two clocks move with velocity v parallel to the line joining them, the clock in the rear will be found to be ahead of the clock in the front by an amount $kv/c^{2.}$
This is the relativity of simultaneity.)

Corollary 4.1 If two events are found in their proper frame to occur simultaneously and to be separated by a distance k, then in an inertial frame moving with velocity v parallel to the line joining the two events, the forward event will occur a time $kv / c^2(1-v^2/c^2)$ after the event in the rear occurs.

Rule 4a As Rule 3 may be viewed as the spatial version of Rule 2, there is strictly a spatial version of Rule 4, which might be termed the "relativity of simullocality". It would read: "Two events which occur in the same place at a time t apart in their proper frame are separated by a distance

$tv / (1-v^2/c^2)$ in a frame in which they move with uniform velocity v ".

Rule 5 If two clocks are synchronised and separated by a distance k in their proper frame, then in an inertial frame in which the two clocks move with velocity v perpendicular to the line joining them, the two clocks are also synchronised.

Corollary 5.1 If two events are found in their proper frame to occur simultaneously and be separated by a distance k, in an inertial frame moving with velocity v perpendicular to the line joining the two events, the two events will also be found to occur simultaneously.

$E = mc^2$

The equation $e = mc^2$ was derived by Einstein in a short paper published after the Special Theory of Relativity but before the General Theory. Here is a quick flavour of its derivation.

Momentum in 3D space is given by

$$mv = m\, \delta x/\delta t \,.$$

To obtain momentum in a 4 dimensional spacetime context we replace δx by δs and δt by $\delta s/c$.

Thus momentum in 4 dimensional spacetime is always mc. That is in 4 dimensional spacetime we are all moving at the speed of light. Either we are stationary in space and all the component is channelled in time or we borrow some from this time dimension to move in space but the total momentum is still conserved. If we borrow all of it and move at velocity c then time "stands still".

Now setting $\gamma = 1 / (1-v^2/c^2)$ we resolve the momentum_{4sd} into a space component and a time component.

The space direction becomes

$$m\delta x/ (\delta t/\gamma) = \gamma\, \delta x / \delta t = \gamma mv.$$

This approximates to the usual mv when v is much less than c.

The time direction becomes mc

$$\delta t / (\delta t/\gamma) = \gamma mc.$$

Now momentum is conserved in spacetime so both γmv and γmc are conserved.

If γmc is conserved then so is γmc^2.

Now we can approximate γ by

$$1 + ½\, v^2/ c^2$$

So γmc^2 is approximately equal to $mc^2 + ½mv^2$ (plus other terms in v).

Now setting $v = 0$ gives the rest mass equation **$e = mc^2$** *Voila!*