Summation $u_r = r(r+k)$

Introduction

I am seeking the general formula to sum expressions such as

$1\times2 + 2\times3 + 3\times4 \ldots$

$1\times3 + 1\times4 + 1\times5 \ldots$

$1\times(1+k) + 2\times(2+k) + 3\times(3+k)$

each to r terms

Method

If u_r can be reconstructed such that

$u_r \quad = f(r) - f(r-1)$ then by telescoping

$\sum u_r \quad = f(r) - f(0)$

Suppose I want to determine $\sum r(r+1)$

then I make my first guess

$g(r) \quad = (r)(r+1)(r+2)$

$g(r-1) = (r-1)(r)(r+1)$

and $g(r) - g(r-1)$

$\quad = r(r+1)(r+2-r+1)$

$\quad = 3r(r+1)$

so I revise my guess to

$f(r) \quad = {}^1/_3\, g(r)$

$\quad = {}^1/_3\,(r)(r+1)(r+2)$

Example

I want to sum

$1\times2 + 2\times3 + 3\times4 + 4\times5$

by the formula

$f(r) \quad = {}^1/_3\,(r)(r+1)(r+2)$

$_{r=1}\sum^{r=4} ur = {}^1/_3\; 4\times5\times6 = 40$ a true result

General Form $\sum r(r+k)$

I'm now ready to tackle the general form $\sum r(r+k)$ and I surmise it will be of the form $r(r+1)(ar+b)$

$f(r) \quad = (r^2 + r)(ar + b)$

$\quad = ar^3 + br^2 + ar^2 + br$

$\quad ar^3 + (a+b)r^2 + br$

$f(r-1) = (r-1)(r)\{a(r-1) + b\}$

$\quad = (r^2 - r)(ar + b - a)$

$\quad = ar^3 + br^2 - ar^2 - ar^2 - br + ar$

$\quad = ar^3 + (b - 2a)r^2 + (a - b)r$

so $f(r) - f(r-1) = 3ar^2 + (2b - a)r$

Try $\sum r(r+1)$

Now if I go back to $\sum r(r+1)$ then I have $r(r+1) = 3ar^2 + (2b - a)r$

equating $r^2 \Rightarrow 1 = 3a \qquad$ so $a = {}^1/_3$

equating $r \Rightarrow 1 = 2b - {}^1/_3 \quad$ so $b = {}^2/_3$

so $f(r) = ar^3 + (a+b)r^2 + br$

$\quad = {}^1/_3\,(r)(r+1)(r+2)$

Try $\sum r(r+2)$

Now I have

$r(r+2) = 3ar^2 + (2b - a)r$

equating $r^2 \Rightarrow 1 = 3a \qquad$ so $a = {}^1/_3$

equating $r \Rightarrow 2 = 2b - a \quad$ so $b = {}^7/_6$

so $f(r) = ar^3 + (a+b)r^2 + br$

$\quad = {}^1/_6\,(r)(r+1)(2r+7)$

Try $\sum r (r + 3)$

Now I have

$r (r + 3) = 3ar^2 + (2b - a)r$

equating $r^2 \Rightarrow 1 = 3a$ so $a = {}^1/_3$

equating $r \Rightarrow 3 = 2b - a$ so $b = {}^{10}/_6$

so $f(r) = ar^3 + (a + b) r^2 + br$

$= {}^1/_3 (r) (r +1) (r + 5)$

Try $\sum r (r + 4)$

Now I have

$r (r + 4) = 3ar^2 + (2b - a)r$

equating $r^2 \Rightarrow 1 = 3a$ so $a = {}^1/_3$

equating $r \Rightarrow 4 = 2b - a$ so $b = {}^{13}/_6$

so $f(r) = ar^3 + (a + b) r^2 + br$

$= {}^1/_6 (r) (r +1) (2r + 13)$

General Form

By starting each expression with ${}^1/_6$ and inserting 2r into the last bracket, the general form is quickly deduced as

$\sum r (r + k)$

$= {}^1/_6 (r) (r +1) (2r + 3k + 1)$

Verification

Finally I demonstrate my new formula with 2 examples

Set r = 4 and k = 5

ie find the sum $_{r=1}\sum^{r=4} u_r$

Now this is $1\times6 + 2\times7 + 3\times8 + 4\times9$

= 80 so I know where I'm heading

By my new formula

$\sum r (r + 5)$ for 4 terms is

$= {}^1/_6 (4) (5) (24) = 80$

Set r = 5 and k = 8

ie find the sum $_{r=1}\sum^{r=4} u_r$

Now this is

$= 1\times9 + 2\times10 + 3\times11 + 4\times12 + 5\times13$

= 175 so I know where I'm heading

By my new formula

$\sum r (r + 5)$ for 5 terms is

$= {}^1/_6 (5) (6) (35) = 175$

Bonus

Given $\sum r (r + k)$

$= {}^1/_6 (r) (r +1) (2r + 3k + 1)$

by setting k = 0 o get

$1^2 + 2^2 + 3^2 + 4^2 \ldots$ for n terms is

$= {}^1/_6 (n) (n +1) (2n + 1)$

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