Theory of Equations

Preliminaries

Assume most questions will focus on a cubic.

$\sum \alpha^2$ means $\alpha^2 + \beta^2 + \gamma^2$

$\sum \alpha^2(\beta + \gamma) =$

$\alpha^2(\beta + \gamma) + \beta^2 (\gamma + \alpha) + \gamma^2 (\alpha + \beta)$

etc.

Introduction

Quadratic

Let $ax^2 + bx + c = 0$

and if the roots are a and b then

$(x - \alpha) (x - \beta) =$

$$x^2 - (\alpha + \beta) x + \alpha\beta = 0$$

and thus $- (\alpha + \beta) = b$ and $\alpha\beta = c$

By "completing the square" it can be shown that

$a = {}^{\{- b + \sqrt{(b^2-4ac)}\}} / {}_{2a}$ and

$b = {}^{\{- b - \sqrt{(b^2-4ac)}\}} / {}_{2a}$

Cubic

Let $ax^3 + bx^2 + cx + d = 0$

$(x - \alpha) (x - \beta) (x - \gamma) =$

$$x^3 - \sum\alpha + \sum\alpha\beta - \alpha\beta\gamma$$

$b = - \sum\alpha \qquad c = \sum\alpha\beta \qquad \gamma = \alpha\beta\gamma$

Higher Order

For higher order algebraic equations the solutions have to come in conjugate pairs if the coefficients are to be real. For odd ordered equation the remaining solution will be real because all such equations must cross the x axis at some point.

Relationship 1

$(\sum\alpha)^2 = \sum\alpha^2 + 2\sum\alpha\beta$

$\sum\alpha^2 = (\sum\alpha)^2 - 2\sum\alpha\beta$

Relationship 2

$\sum\alpha \times \sum\alpha\beta$

$= 3\alpha\beta\chi + \sum\alpha^2 (\beta + \gamma)$

$\sum\alpha^2 (\beta + \gamma) = \sum\alpha \times \sum\alpha\beta - 3\alpha\beta\chi$

Relationship 3

$\sum\alpha^2 \times\sum\alpha = \sum\alpha^3 + \sum\alpha^2 (\beta + \gamma)$

$\sum\alpha^3 = \sum\alpha^2 \times\sum\alpha - \sum\alpha^2 (\beta + \gamma)$

and we have already determined

$\sum\alpha^2 (\beta + \gamma)$ in relationship 2.

Relationship 4

$(\sum\alpha\beta)^2 = \sum\alpha^2\beta^2 + 2 \sum\alpha \times \alpha\beta\gamma$

$\sum\alpha^2\beta^2 = (\sum\alpha\beta)^2 - 2 \sum\alpha \times \alpha\beta\gamma$

Example 1

let $f(x) = x^3 + 5x^2 + 2x + 3$ so

$- \sum\alpha = 5 \qquad \sum\alpha\beta = 2 \qquad - \alpha\beta\gamma = 3$

i) Find $\sum\alpha^2$

From **R1** $\sum\alpha^2 \quad = (\sum\alpha)^2 - 2\sum\alpha\beta$

$(-5)^2 - 2 (2) = 21$

ii) Find $\sum\alpha^2(\beta+\gamma)$

From **R2**

$\sum\alpha^2(\beta+\gamma) = \sum\alpha \times \sum\alpha\beta - 3\alpha\beta\chi$

$= (-5)(2) - (-3) = -1$

iii) Find $\sum\alpha^3 =$

$\sum\alpha^3 = \sum\alpha^2 \times \sum\alpha - \sum\alpha^2(\beta+\gamma)$

$= (21)(-5) - (-1)$

$= 104$

iv) Find $\sum\alpha^2\beta^2 =$

From **R4**

$\sum\alpha^2\beta^2 = (\sum\alpha\beta)^2 - 2\sum\alpha \times \alpha\beta\gamma$

$= (2)^2 - 2(-5)(-3) = -26$

Example 2

$x^3 + 5x^2 - 2x + 3$ has roots $\alpha\ \beta\ \gamma$

i) Find the equation with roots

$\alpha + 1 \quad \beta + 1 \quad \gamma + 1$

if x takes $\alpha\ \beta\ \gamma$

then y takes $\alpha + 1 \quad \beta + 1 \quad \gamma + 1$

$x^3 + 5x^2 - 2x + 3 \Rightarrow$

$(y-1)^3 + 5(y-1)^2 - 2(y-1) + 3$

$= (y^3 - 3y^2 + 3y - 1) + 5(y^2 - 2y + 1)$

$+ (2y - 2) + 3$

which multiplies out to $y^3 + 2y^2 - 5y + 5$

ii) Find the equation with roots

$3\alpha \quad 3\beta \quad 3\gamma$

We can adopt exactly the same procedure

$x^3 + 5x^2 - 2x + 3 \Rightarrow$

$({}^y/_3)^3 + 5({}^y/_3)^2 - 2({}^y/_3) + 3$

which if you care to check out

$\Rightarrow y^3 + 15y^2 + 18y + 81$

How would we check out our answer.

There seems not an easy way particularly given two of the roots of the given function are complex.

iii) Find the equation with roots

$\alpha^2 \quad \beta^2 \quad \gamma^2$

The clue is to generate the expressions

$(x^2 - \alpha^2) \quad (x^2 - \beta^2) \quad (x^2 - \gamma^2)$

and for example

$(x^2 - \alpha^2) = (x - \alpha)(x + \alpha)$

We know

$(x - \alpha)(x - \beta)(x - \gamma) =$

$x^3 + 5x^2 + 2x + 3$

and keeping careful track of sign changes

$(x + \alpha)(x + \beta)(x + \gamma) =$

$x^3 - 5x^2 + 2x - 3$

Multiply left and right and sides

$(x^2 - \alpha^2)(x^2 - \beta^2)(x^2 - \gamma^2) =$

$(x^3 + 5x^2 + 2x + 3)(x^3 - 5x^2 + 2x - 3)$

Let $A = x^3 + 2x$ and $B = 5x^2 + 3$

so we have $(A + B)(A - B) = A^2 + B^2$

$(x^3 + 5x^2 + 2x + 3)(x^3 - 5x^2 + 2x - 3) =$

$(x^3 + 2x)^2 - (5x^2 + 3)^2$

Finally let $y = x^2$ so expression becomes

$(x^3 + 2x)^2 = \{x(x^2 + 2)\}^2$

$= x^2(x^2 + 2)^2 = y(y + 2)^2$

and similarly $(5x^2 + 3)^2 = (5y + 3)^2$

Multiplying all that out and remembering signs final expression is

$y^3 - 21y^2 - 26y + 9$

Would I be able to knock that out under exam conditions? Mmmmm.

rg

Investigation into inverse roots

Start with $(x-1)(x-2)(x-3)$
$= x^3 - 6x^2 + 11x - 6$
This enables me to check out the answer is correct.

What equation has roots $^1/_\alpha\ ^1/_\beta\ ^1/_\gamma$
So do we just replace x with $^1/_y$?
$x^3 - 6x^2 + 11x - 6 = 0 \Rightarrow$
$(^1/_y)^3 - 6(^1/_y)^2 + 11(^1/_y) - 6 = 0 \Rightarrow$
$(^1/_{y^3})^3 - 6(^1/_{y^2})^2 + 11(^1/_y) - 6 = 0 \Rightarrow$
$1 - 6y + 11y^2 - 6y^3 = 0.$

However I need to check if this is correct.

$(x-1)(x-½)(x-^1/_3) = 0 \Rightarrow$
$y^3 - 11y^2 + 6y - 1 = 0 \Rightarrow$
$1 - 6y + 11y^2 - 6y^3 = 0$

So the direct substitution works.