How long is the paper in a toilet roll?

About fifteen years ago I was at a conference. The speaker was sufficiently competent to be worth taping. It was a reel-to-reel and as I watched in turn slowly round I thought to myself "How could I calculate how long the tape is?". Strangely the problem cropped up again in our department but this time it was the length of toilet rolls.

Let the inner radius (ie the tube) = r
Let outer radius (the whole roll) = R
Let with thickness = t.
1^{st} wrap-around is at radius $r + {}^{1}/_{2}\, t$,
2^{nd} wrap-around is at radius $r + {}^{3}/_{2}\, t$
Last wrap is at radius $R - {}^{1}/_{2}\, t$.
There are n wrap-arounds $n = {}^{(R-r)} / {}_{t}$
Length of each wrap is radius $\times\, 2\pi$,
So total length $= 2\pi \{ (r + {}^{1}/_{2}\, t) + (r + {}^{3}/_{2}\, t) + \ldots \}$ for n terms
$= 2\pi n r + \pi t\, (1 + 3 + 5 + 7 + \ldots)$ again for n terms.

Summing $(1 + 3 + 5 + 7 \ldots n)$ is just n^2.
Hence total length $= 2\, \pi\, n\, r + \pi\, t\, n^2$

Substitute back $n = {}^{(R-r)} / {}_{t}$ and multiply the whole lot out you'll discover everything conveniently cancels out and

Total length $= {}^{\pi}/_{t}\, (R^2 - r^2)$
which looks worryingly familiar.

There is a story of Fourier who spent over a year multiplying out a particularly complicated expression, making three fundamental errors along the way, which fortunately proved inconsequential, and when he got down to the final answer realised he could have done the whole thing in two lines. Worse, Euler had beaten him to it by 30 years. Still at least posterity records it as "Fourier Analysis".

Or more down to earth, take my old Maths teacher (a Welshman so you must think of the next bit in a Welsh accent) who would have said *"Excellent Goodhand but it's all a bit Haverfordwest to get to Neyland isn't it?"*
I didn't have a clue what he was talking about until I mentioned it to my dad who had the insight to get a map out and then the penny dropped. Can you do the same? (Hint : he grew up in Pembroke. And it's no good thinking about it – you've got to get a map out and look.)
So here's another way.

Take the end area of the toilet roll as $\pi\, (R^2 - r^2)$. Divide that by the thickness t and you have the length ${}^{\pi}/_{t}\, (R^2 - r^2)$. Not even two lines, just one.

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