A Narrative to understand the 24 Trigonometric Functions

Summary

There are 24 trig functions in total

12 Circular Functions

sin cos tan sec cosec cot

$\sin^{-1}$ $\cos^{-1}$ $\tan^{-1}$ $\sec^{-1}$ cosec^{-1} $\cot^{-1}$

12 Hyperbolic Functions

sinh cosh tanh sech cosech coth

$\sinh^{-1}$ $\cosh^{-1}$ $\tanh^{-1}$ sech^{-1} cosech^{-1} $\coth^{-1}$

This explanatory sheet gives a clear narrative on how these functions arise and deduces in an intuitive way how the series for the six main functions arise. Finally it gives the relationship between the circular functions and the hyperbolic functions.

Introduction

First we need to define some terms

Reciprocal and Inverse

For a number k the reciprocal is $^1/_k$

The inverse of ×k *(times k)* is ÷k

The inverse of +k is –k

So for a pure number it is important to understand that the **reciprocal** of a number is its **multiplicative inverse,** while the **negation** of a number is its **additive inverse**.

But now consider the function f(x).

Suppose f(x) = 5x + 4. The inverse of this function is denoted by $f^{-1} = {}^{(x-4)}/_5$.

So if $f(3) = 5 \times 3 + 4 = 19$

then $f^{-1}(x) = {}^{(19-4)}/_5 = 3$

To be clear while $k^{-1} = {}^1/_k$

and the reciprocal of f(x) is $^1/_{f(x)}$

$f^{-1}(x)$ is the inverse of f(x). It is the function that undoes the effect of the original function. That $f^{-1}\{ f(x) \} = x$

To be clear we've "borrowed" the symbol "**–1**" correct for reciprocals of numbers which have an inverse connotation to mean inverses for functions.

Consequently we use the same symbol to denote the inverse of trig functions.

So for the sine function

sin θ takes an angle input θ

and outputs a number x while

$\sin^{-1}(x)$ takes a number x

and outputs an angle θ. [#1]

It is only necessary to emphasise

$\sin^{-1}(x)$ is the inverse of sin (x)

$\sin(x^{-1})$ is the sine of $(^1/_x)$

$(\sin x)^{-1}$ is $^1/_{\sin x}$

and if you come across $\sin(x)^{-1}$ then send it back because "*who knows?*".

Terminology

The term $\sin^{-1} x$ is best pronounced *arcsine x* and if looking up information on the internet search *arcsine*. The author discourages writing arcsin x because every calculator button is clearly labelled $\sin^{-1}$ x.

Trigonometric Functions

Strictly the term covers two groups of functions – the **circular functions** sine, its complement **co**sine and the ratio of the two - tangent - plus their inverses and reciprocals and also the **hyperbolic functions** sinh *(shine)* cosh *(cosh)* and tanh *(tan h)* plus their inverses and reciprocals. So in total we have 24 trigonometric functions

Circular Functions

sin	sine
cos	cosine
tan	tangent

Reciprocals

csc	cosecant
sec	secant
cot	cotangent

Inverses

The prefix "arc" is used as the function relates to the arc length of a unit circle

$\sin^{-1}$	arcsine
$\cos^{-1}$	arccosine
$\cot^{-1}$	arccotangent

Reciprocals of Inverses

$\csc^{-1}$	arccosecant
$\sec^{-1}$	arcsecant
$\cot^{-1}$	arccotangent

Hyperbolic Functions

sinh	"shine"
cosh	"cosh"
tanh	"tan h"

Reciprocals

csch	"cosec h"
sech	"sec h"
coth	"cot h"

Inverses

The prefix "**ar**" *(stet)* is used as the function relates to the area enclosed by a hyperbola.

$\sinh^{-1}$	"arshine"
$\cosh^{-1}$	"arcosh"
$\coth^{-1}$	"arcot h"

Reciprocals of Inverses

csch^{-1}	"arcosec h"	
sech^{-1}	"arsec h"	
$\coth^{-1}$	"arcot h"	#2

Relationship Circ. Hyp. Functions

Preliminaries

Now any function can be split into an even function and an odd function where

$f_e(x) = ½ \{f(x) + f(–x)\}$ and

$f_0(x) = ½ \{f(x) – f(–x)\}$.

Even functions are made up of the even powers of x and odd functions are made up of the odd powers of x. Even functions have symmetry about the y-axis and odd functions have rotational symmetry 2.

Now as e^x can be represented by an infinite algebraic polynomial it can be split into an even function and an odd function.

$e^x = f_{even}(x) + f_{odd}(x)$ where

$f_{even}(x) = \cosh(x)$ - the "even" terms

and $f_{odd}(x) = \sinh(x)$ - the "odd" terms

By transforming $x \to ix$ we have

$e^{ix} = \cosh(ix) + \sinh(ix)$

Now knowing the expansion of e^x is

$1 + x^0/0! + x^1/1! + x^2/2! + x^3/3! \ldots$

consider the effect of replacing x with ix.

The terms x^2, x^6, x^{10} etc. $x \to (-x)$ $i^2 = -1$

The terms x^4, x^8, x^{12} etc. $x \to (x)$ $i^4 = 1$

The terms x^1, x^5, x^9 etc. $x \to (ix)$

The terms x^3, x^7, x^{11} etc. $x \to (-ix)$

Therefore it is clear that

$e^{ix} = f_{even}(ix) + f_{odd}(ix)$

and $f_{even}(ix)$ has no terms in "i"

and $f_{odd}(ix)$ has all terms in "i"

but both series alternate +ve / –ve

Now amazingly it transpires that

$f_{even}(ix) = \cos x$

$f_{odd}(ix) = i \sin x$

Combining these results we produce Euler's famous relationship

$e^{ix} = \cos x + i \sin x$ and comparing to

$e^{ix} = \cosh ix + \sinh ix$

we immediately deduce the key relationships $\cosh ix = \cos x$ and

$\sinh ix = i \sin x$

If you've understood thus far you will never forget.

Specifically the series are

$\cos x = x^0/0! - x^2/2! + x^4/4! - x^6/6! \ldots$

remembering $x^0/0! = 1$

$\sin x = x^1/1! - x^3/3! + x^5/5! - x^7/7! \ldots$

remembering $x^1/1! = x$

$\cosh x = x^0/0! + x^2/2! + x^4/4! + x^6/6! \ldots$

remembering again $x^0/0! = 1$ and

$\sinh x = x^1/1! + x^3/3! + x^5/5! + x^7/7! \ldots$

remembering again $x^1/1! = x$

Investigations

It is instructive when given any transformation of one trigonometric function say $^d/_{dx} \sin x = \cos x$ then to investigate the effect on the other 23 functions. You will invariably discover a fascinating interplay of symmetry and asymmetry but often just one of the 24 transformations will refuse stubbornly to fit into an overall pattern.

Logarithmic Representations

Because the hyperbolic functions relate to the exponential function they can be expressed as a natural logs for example

$\sinh^{-1} x = \ln \{x + \sqrt{(x^2 + 1)}\}$

though the derivation takes a few lines. Perhaps more surprisingly but through the same reasoning inverse circular functions can be expressed as complex logs eg.

$\sin^{-1} x = -i \ln \{ix \pm \sqrt{(1 - x^2)}\}$ $^-1 \le x \le {}^+1$

Calculators

Accessing all these functions on a calculator is an exercise in itself. All calculators are a compromise of multiplicity of buttons competing with an uncluttered look.

Casio Range (KS4)

The three key functions sin cos tan have their own keys with the inverses located above plus a “hyp” button for the corresponding hyperbolic functions.

The inverse of the functions sin cos tan, sec cosec cot, can be calculated directly.

So sin 30 = 0.5 and cosec 30 = ${}^{1}/_{\sin 30}$ = 2.

However when we try to find the values of say $\csc^{-1}$ x we to hit a problem.

Clearly $\csc^{-1} 2 = 30°$ but entering $1/\sin^{-1} 2$ gives Ma ERROR. Here’s the trick.

Let	sin b	$= {}^{1}/_{a}$
Then	b	$= \sin^{-1} ({}^{1}/_{a})$
However	csc b	= a
so	b	$= \csc^{-1} a$

equating the two results

$$\csc^{-1} a = \sin^{-1} ({}^{1}/_{a})$$

so checking $\csc^{-1} 2 = \sin^{-1} ({}^{1}/_{2})$

and our calculator dutifully gives the answer 30° or ${}^{\pi}/_{6}$ if checking on the internet.

Texas Graphical Calculator

The Texas TI range is much less cooperative. The hyperbolic functions can only be accessed through the CATALOG button – not particularly helpful.

Footnotes

#1 The concept of $\sin^{-1}$ is much easier to understand because its use is immediately apparent - given the ratio of two sides of a right angled triangle you can immediately deduce the angle. The purpose of sin x is more obscure.

#2 Searching **arcsinh** *(stet)* on Google produces 135 000 erroneous “hits” including the programming language ***“Wolfram Language”***. The writers really ought to know better.

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